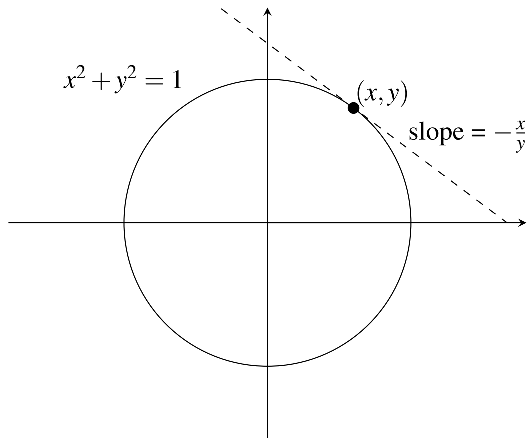
§2.6 Implicit Differentiation

Thus far, we have mostly computed derivatives $\frac{\mathrm{d}y}{\mathrm{d}x}$ by using *explicit* formulas $y = f(x)$. By using the chain rule, it is possible to $\frac{\mathrm{d}y}{\mathrm{d}x}$ when y is only *implicitly* a function of x.

Example: The equation $x^2 + y^2 = 1$ defines the circle of radius 1 centered at $(0, 0)$.

"Explicit" differentation: Write y (explicitly) as a function of x and differentiate:

$$y = \pm\sqrt{1 - x^2}$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mp x}{\sqrt{1 - x^2}}$$



Implicit differentiation: Assume y is implicitly a function of x, and use the chain rule to differentiate both sides of $x^2 + y^2 = 1$ with respect to x:

$$2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}, \quad y \neq 0.$$

Notes:

- When we use implicit differentiation to find $\frac{\mathrm{d}y}{\mathrm{d}x}$ we assume that (a piece of) the curve can be written in the form $y = f(x)$. Sometimes this is not possible, for example a small piece around $(1, 0)$ or $(-1, 0)$ of the above curve.
- Given an equation relating x and y, we can also try to express x as a function of y, and compute $\frac{\mathrm{d}x}{\mathrm{d}y}$ using the same method. As one may guess, we have $\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{1}{(\frac{\mathrm{d}y}{\mathrm{d}x})}$ when $\frac{\mathrm{d}y}{\mathrm{d}x} \neq 0$. We say there is a vertical tangent line when $\frac{\mathrm{d}x}{\mathrm{d}y} = 0$.
  - e.g. If $x^2 + y^2 = 1$ then $\frac{\mathrm{d}x}{\mathrm{d}y} = -\frac{y}{x}$ for $x \neq 0$, and we see that there is a vertical tangent line whenever $y = 0$, which forces $x^2 = 1$, hence at $(1, 0)$ and $(-1, 0)$. Notice that while we can't write $y = f(x)$ near these points, we can write $x = g(y)$.
- Implicit differentiation is especially useful for finding $\frac{\mathrm{d}y}{\mathrm{d}x}$ when it is inconvenient to write y as a function of x. It only requires an equation relating x and y to differentiate.
  - e.g. We can not solve $ye^y = xe^x$ for y in terms of standard functions,but we can differentiate both sides and solve to find $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{(x+1)e^x}{(y+1)e^y}$.
- When we allow the expression we find for $\frac{\mathrm{d}y}{\mathrm{d}x}$ to involve both x and y, because there is an assumed relationship between x and y, this expression will not be unique.
  - e.g. If $x^2 = y^3 + y^2$ then $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{2x}{3y^2 + 2y} = \dfrac{2x}{3x^2 - 3y^3 + 2y}$.

  The second expression is obtained by replacing y^2 in the denominator of the first expression by $x^2 - y^3$, which is true at any point on the curve.

Exercises

1. Use implicit differentiation to find a formula for $\frac{dy}{dx}$ along each curve below.

$$x^2 + y^2 = 2 \qquad x^2 + 4y^2 = 4 \qquad y^2 = x^3 - x + 1$$
$$xy = 1 \qquad xy = x^2 + y^2 - 1 \qquad \tan(xy) = \frac{2x}{x^2 - 1}$$

2. Find the tangent line to each curve at the point indicated.

$$x = y^2, \quad (4, -2) \qquad 2x^2 + 2xy + y^2 = 1, \quad (\tfrac{1}{\sqrt{2}}, 0)$$
$$x^2 + y^2 = 5, \quad (3, -4) \qquad y^2 = x^3 - x + 1, \quad (3, 5)$$
$$ye^y = x^2 e^x, \quad (1, 1) \qquad x^2 - 2xy - x + y^2 - y = 0, \quad (-\tfrac{1}{8}, \tfrac{3}{8})$$

3. Find all points on the given curve with either a vertical or horizontal tangent line.

$$x^2 + y^2 = 1 \qquad x^2 - xy + y^2 = 1$$
$$y^2 = x^3 - x + 1 \qquad 4xy\left(x^2 - y^2\right) = 2\left(x^2 + y^2\right)$$

4. For each value of m, find all points on the given curve with slope m.

$$x^2 + y^2 = 1, \quad m = \frac{3}{4}, \quad m = -\frac{12}{5}, \quad m = \frac{65}{72}$$
$$x^2 - 2xy - x + y^2 - y = 0, \quad m = -1, \quad m = 1, \quad m = \frac{1}{3}, \quad m = 3$$