The World's Sneakiest Substitution (optional)

The "world's sneakiest substitution" is $m = \tan(\frac{t}{2})$, which leads to the formulas

$$\cos(t) = \frac{1-m^2}{1+m^2}, \quad \sin(t) = \frac{2m}{1+m^2}, \quad \mathrm{d}t = \frac{2}{1+m^2}\,\mathrm{d}m.$$

The expression for $\mathrm{d}t$ comes from

$$\mathrm{d}m = \tfrac{1}{2}\sec^2(\tfrac{t}{2})\,\mathrm{d}t = \tfrac{1}{2}(1+\tan^2(\tfrac{t}{2}))\,\mathrm{d}t = \tfrac{1}{2}(1+m^2)\,\mathrm{d}t,$$

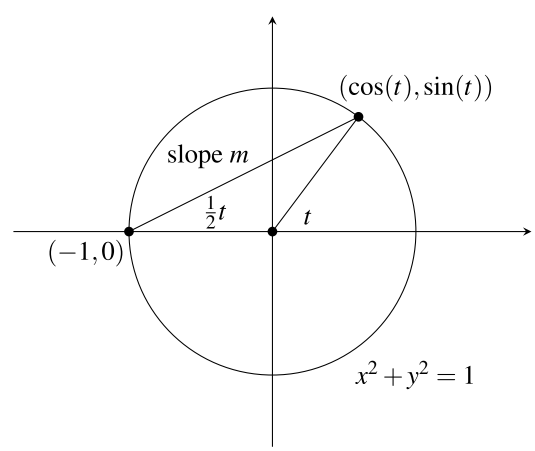
but the remarkable part is that $\cos(t)$ and $\sin(t)$ are converted to rational expressions in m. Thus, by this change of variables, an integral involving the sum, product, or quotient of factors of $\sin(t)$ or $\cos(t)$ in an integral are converted to rational expressions.

Example: The integral $\int \frac{1}{\sin(t)+1}\,\mathrm{d}t$ is quite intractable by other methods, but the above substitution gives

$$\int \frac{1}{\sin(t)+1}\,\mathrm{d}t = \int \frac{1}{\left(\frac{2m}{m^2+1}\right)+1}\cdot\frac{2}{m^2+1}\,\mathrm{d}m = \int \frac{2}{m^2+2m+1}\,\mathrm{d}m =$$
$$= \int \frac{2}{(m+1)^2}\,\mathrm{d}m = \frac{-2}{m+1} + C = \frac{-2}{\tan(\frac{t}{2})+1} + C.$$

Rational Parameterization of the Circle

One could view the substitution as a trick, and prove the above formulas with trigonometric identities. Howeve, another way to think about it is that the substitution comes from parameterizing the unit circle via the slope of a line through $(-1,0)$, as pictured below. The expressions for $\cos(t)$ and $\sin(t)$ come from finding the intersection of the line $y = m(x-1)$ and the circle $x^2+y^2=1$.



Application: Pythagorean Triples

The fact that $\left(\frac{1-m^2}{1+m^2}\right)^2 + \left(\frac{2m}{1+m^2}\right)^2 = 1$ for all m can lead to interesting identities when m is a rational number. For example if we set $m = \frac{2}{7}$, we get

$$1 = \left(\frac{1-(\frac{2}{7})^2}{1+(\frac{2}{7})^2}\right)^2 + \left(\frac{2(\frac{2}{7})}{1+(\frac{2}{7})^2}\right)^2 = \left(\frac{45}{53}\right)^2 + \left(\frac{28}{53}\right)^2 \implies 28^2+45^2=53^2.$$

In fact, all the Pythagorean triples can be found in this way.

Application: Integrals of $\sec(t)$, $\csc(t)$

We can apply the substitution to integrate $\csc(t)$ as follows:

$$\int \csc(t)\,\mathrm{d}t = \int \frac{1}{\sin(t)}\,\mathrm{d}t = \int \frac{1+m^2}{2m}\cdot\frac{2}{1+m^2}\,\mathrm{d}m = \int \frac{1}{m}\,\mathrm{d}m = \ln|m| + C = \ln\left|\tan\left(\tfrac{t}{2}\right)\right| + C.$$

Because of the many algebraic identities for trigonometric and logarithmic functions, the above can be represented in a number of ways. For example, we have

$$\tan\left(\tfrac{t}{2}\right) = \frac{\sin(t)}{1+\cos(t)} = \frac{1-\cos(t)}{\sin(t)} = \frac{1}{\csc(t)+\cot(t)} = \csc(t) - \cot(t).$$

The version in the book's table of integrals comes from the last expression:

$$\boxed{\int \csc(t)\,\mathrm{d}t = \ln|\csc(t) - \cot(t)| + C}$$

The analogous computation for $\sec(t)$ requires either trigonometric substitutions (§6.2) or partial fraction decomposition (§6.3), but we can also use the fact that $\cos(t) = \sin(t + \frac{\pi}{2})$, hence $\sec(t) = \csc(t + \frac{\pi}{2})$ to compute

$$\int \sec(t)\,\mathrm{d}t = \int \csc(t + \tfrac{\pi}{2})\,\mathrm{d}t = \ln|\csc(\tfrac{\pi}{2} + t) - \cot(\tfrac{\pi}{2} + t)| + C.$$

Since $\cot(\frac{\pi}{2} + t) = \dfrac{\cos(\frac{\pi}{2} + t)}{\sin(\frac{\pi}{2} + t)} = \dfrac{-\sin(t)}{\cos(t)} = -\tan(t)$, we obtain the textbook's form:

$$\boxed{\int \sec(t)\,\mathrm{d}t = \ln|\sec(t) + \tan(t)| + C.}$$