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Stationary black hole metrics and inverse problems in two space dimensions

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Abstract

We study the wave equation for a stationary Lorentzian metric in the case of two space dimensions. Assuming that the metric has a singularity of the appropriate form, surrounded by an ergosphere which is a smooth Jordan curve, we prove the existence of a black hole with a boundary (called the event horizon) that is piecewise smooth, generally having corners. We consider a physical model of acoustic black hole whose event horizon has corners. Finally we consider the determination of a black hole by the boundary measurements.

Keywords: black hole, acoustic metric, ergosphere

(Some figures may appear in colour only in the online journal)

1. Introduction

Consider the wave equation associated to a stationary metric on $\mathbb{R}^{1+2} \cong \mathbb{R}_{x_0}^1 \times \mathbb{R}_{(x_1, x_2)}^2$,

$$\sum_{i,j=0}^2 \frac{1}{\sqrt{g(x)}} \frac{\partial}{\partial x_i} \left(\sqrt{g(x)} g^{ij}(x) \frac{\partial u(x_0, x)}{\partial x_j} \right) = 0, \quad (x_0, x) \in \mathbb{R}^{1+2}. \quad (1.1)$$

Here, $[g^{ij}(x)]_{i,j=0}^2$ is the inverse of $[g_{ij}(x)]_{i,j=0}^2$, where $g_{ij}(x) \in C^\infty(\mathbb{R}^{1+2}; \mathbb{R})$ defines a pseudo-Riemannian metric with signature $(+1, -1, -1)$ depending only on x , with $g_{ij}(x) = g_{ji}(x)$, and $g(x) = \det[g_{ij}(x)]_{i,j=0}^2$.

For some choices of $g^{jk}(x)$, equation (1.1) has a black hole, i.e. a region which disturbances may not propagate out of. These are often called *analogue* or *artificial* black holes, since the metric is in general not a solution of the Einstein equations of general relativity [FN98, Wal10] (A precise definition of black and white holes is given below).

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Two of the most common examples arising from physical models are *optical black holes* (see [BCO⁺11, Gor23, LP00, PKR⁺08]) and *acoustic black holes* (see [Unr81, Vis98]). In optics, equation (1.1) is a model for the propagation of light through an inhomogeneous moving medium, while in acoustics, it models the propagation of acoustic waves in a moving fluid. Physicists are interested in physical systems which may contain analogue black holes, as they may be suitable for experimental study, while providing some insight into phenomena of general relativity. A number of other models have been studied, including surface waves, relativistic acoustic waves, Bose–Einstein condensates and others [FFL⁺10, RMM⁺10, SU02, VMP10]. See [BLV⁺05, NVV02, Vis12] for surveys and many references.

We define an *event horizon* for (1.1) to be a Jordan curve $S_0 \subseteq \mathbb{R}^2$ such that $\mathbb{R} \times S_0$ is piecewise characteristic and forward null-geodesics either cannot pass from the interior to the exterior of S_0 , or vice versa. See section 2. In the former case we will say that the region enclosed by $\mathbb{R} \times S_0$ is a *black hole*, and in the latter case we call it a *white hole*.

Let $O = (0, 0)$ be a singularity of the metric and assume that g^{jk} behaves near O as in [Esk14]: When $|x| < \varepsilon$, assume that

$$g^{jk}(x) = g_1^{jk}(x) + g_2^{jk}(x), \quad (1.2)$$

where g_1^{jk} is similar to an acoustic metric (see also section 4):

$$g_1^{00} = 0, \quad g_1^{j0} = g_1^{0j} = v^j, \quad j = 1, 2, \quad g_1^{jk} = v^j v^k, \quad j, k = 1, 2, \quad (1.3)$$

where in polar coordinates $x_1 = r \cos \theta$, $x_2 = r \sin \theta$, $\hat{r} = \left(\frac{x_1}{r}, \frac{x_2}{r}\right)$, $\hat{\theta} = \left(-\frac{x_2}{r}, \frac{x_1}{r}\right)$, we have

$$v = (v^1, v^2) = \frac{b_1}{r} \hat{r} + \frac{b_2}{r} \hat{\theta}, \quad (1.4)$$

where $b_j = b_j(\theta)$, $j = 1, 2$ are smooth with $b_1(\theta) \neq 0$. Assume also that g_2^{jk} is smooth in (r, θ) , with $g_2^{00} \geq C > 0$, $g_2^{j0} = g_2^{0j} = \mathcal{O}(r)$, $1 \leq j \leq 2$, and $[g_2^{jk}]_{j,k=1}^2$ a negative definite matrix when $|x| < \varepsilon$:

$$([g_2^{jk}]_{j,k=1}^2 \alpha, \alpha) \leq -C_0 |\alpha|^2, \quad \alpha \in \mathbb{R}^2.$$

Let $\Delta(x) = g^{11}(x)g^{22}(x) - (g^{12}(x))^2$. Define the *ergoregion* to be the set $\Omega \subseteq \mathbb{R}^2$ where $g_{00}(x) < 0$. By the Cramer rule $g_{00}(x) = \Delta(x)g(x)$ with $g(x)$ as in (1.1). Thus $\Delta(x) < 0$ is equivalent to $g_{00}(x) < 0$. Assume the boundary $\partial\Omega = \{\Delta(x) = 0\}$, called the *ergosphere*, is a Jordan curve encircling O that is smooth in the sense that the gradient of $\Delta(x)$ is nonzero on $\partial\Omega$.

In [Esk10], it was shown that if the ergosphere is a smooth characteristic surface or non-characteristic surface which contains a trapped surface, then it contains a black hole or a white hole. See also [Esk14] and [Hal13]. In this paper we prove the following much more general result:

Theorem 1.1. *Let g be any metric such that the ergosphere $\{\Delta(x) = 0\}$ for equation (1.1) is a Jordan curve which is smooth in the sense that $\frac{\partial\Delta(x)}{\partial x} \neq 0$ when $\Delta(x) = 0$, and the ergoregion $\Omega = \{\Delta(x) < 0\}$ contains a singularity O , which satisfies (1.2)–(1.4). Then there exists a black hole in $\mathbb{R} \times \bar{\Omega}$ if $b_1(\theta) < 0$, and there exists a white hole if $b_1(\theta) > 0$. Moreover, the event horizon may have corner points, while it is continuously differentiable outside these corner points.*

The plan of the paper is as follows. In section 2, we discuss the general behavior of null geodesics for metrics satisfying the hypotheses of theorem 1.1. In section 3, we prove the existence of a black or white hole and show that the event horizon is C^1 , except at corner points. In section 4, we study acoustic black holes and demonstrate that the event horizon may

have corners. In section 5 we study the determination of a black hole’s horizon by the boundary measurements on $\mathbb{R} \times \mathcal{D}$ where $\mathcal{D} \subset \mathbb{R}^n$ is a bounded domain containing an ergoregion Ω . Assuming that the conditions of theorem 1.1 are satisfied and no point of $\partial\Omega$ is characteristic we prove that the boundary measurements determine the black hole’s event horizon inside Ω up to a change of variables.

2. Null geodesics

2.1. Zero-energy null geodesics in the ergoregion

Consider bicharacteristics for the wave equation (1.1),

$$\frac{dx_p}{ds} = 2 \sum_{k=0}^2 g^{pk}(x(s))\xi_k(s), \quad \frac{d\xi_p}{ds} = - \sum_{j,k=0}^2 g_{x_p}^{jk}(x(s))\xi_j(s)\xi_k(s), \quad 0 \leq p \leq 2.$$

Since the metric is stationary we have that $\xi_0(s)$ is constant. Consider null-bicharacteristics with $\xi_0(s) = 0$. We shall call null-bicharacteristics with $\xi_0(s) = 0$ ‘zero-energy’ null-bicharacteristics. Their projections onto (x_1, x_2) will be called zero-energy null-geodesics. For all s , $x = x(s)$ and $(\xi_1, \xi_2) = \xi = \xi(s)$ must satisfy

$$\sum_{j,k=1}^2 g^{jk}(x)\xi_j\xi_k = 0, \quad (\xi_1, \xi_2) \neq (0, 0), \quad x \in \Omega. \tag{2.1}$$

For each $x \in \Omega$ there are two linearly independent solutions $\xi^\pm = (\xi_1^\pm, \xi_2^\pm)$ of (2.1). It was shown in [Esk10], for $|x| > \varepsilon$, and in [Esk14], for $|x| < \varepsilon$, that there exists a pair of continuous vector fields $f^\pm(x) = (f_1^\pm(x), f_2^\pm(x))$ on $\bar{\Omega} \setminus \mathcal{O}$, satisfying

$$\begin{aligned} 0 &\neq f^+(x) = f^-(x), \quad x \in \partial\Omega, \\ f^+(x), f^-(x) &\text{linearly independent, } \quad x \in \Omega \setminus \mathcal{O}, \\ f_1^\pm(x)\xi_1^\pm + f_2^\pm(x)\xi_2^\pm &= 0, \quad (\xi_1^\pm, \xi_2^\pm) \text{ solving (2.1)}. \end{aligned} \tag{2.2}$$

The choice of sign is arbitrary, but the pair $f^\pm(x)$ is otherwise well-defined up to rescalings which respect (2.2). If we parameterize zero-energy null-bicharacteristics $(x^\pm(x_0), \xi^\pm(x_0))$ by x_0 , then we have

$$\frac{dx_j^\pm}{dx_0} = \frac{g^{j1}(x(x_0))\xi_1^\pm(x_0) + g^{j2}(x(x_0))\xi_2^\pm(x_0)}{g^{01}(x(x_0))\xi_1^\pm(x_0) + g^{02}(x(x_0))\xi_2^\pm(x_0)}, \quad j = 1, 2. \tag{2.3}$$

Since $f_1^\pm(x(x_0))\xi_1^\pm(x_0) + f_2^\pm(x(x_0))\xi_2^\pm(x_0) = 0$ we have that $\xi_1^\pm(x_0) = f_2^\pm(x(x_0))\xi_2^\pm(x_0) = -f_1^\pm(x(x_0))$ up to a nonzero factor. Substituting into (2.3), we get

$$\frac{dx_j^\pm}{dx_0} = \frac{g^{j1}f_2^\pm(x) - g^{j2}f_1^\pm(x)}{g^{10}f_2^\pm(x) - g^{20}f_1^\pm(x)}, \quad j = 1, 2. \tag{2.4}$$

In other words, *the zero-energy null-geodesics in $\Omega \setminus \mathcal{O}$, are the solutions $x = x^+(x_0)$, $x = x^-(x_0)$ of an autonomous system of differential equations.* We shall call the two families of solution curves or trajectories for (2.4) the (+), (−) families respectively.

Note that

$$\frac{dx_2^\pm}{dx_1^\pm} = \frac{g^{21}f_2^\pm - g^{22}f_1^\pm}{g^{11}f_2^\pm - g^{12}f_1^\pm} = \frac{f_2^\pm}{f_1^\pm} \tag{2.5}$$

since $g^{21}f_2^\pm f_1^\pm - g^{22}(f_1^\pm)^2 = g^{11}(f_2^\pm)^2 - g^{21}f_1^\pm f_2^\pm$. Since the rank of $[g^{jk}(x)]_{j,k=1}^2$ is equal to 1 in $\partial\Omega$ we get $\frac{dx_j^\pm}{dx_0} = 0$, $j = 1, 2$, on $\partial\Omega$, but $\frac{dx_2^\pm}{dx_1^\pm}$ has a limit on $\partial\Omega$. Note also that [Esk10]

$$g^{10}f_2^\pm - g^{20}f_1^\pm \neq 0. \quad (2.6)$$

As in [Esk10], we have $f^\pm(x) \cdot \nabla G^\pm(x) = 0$, where $G^\pm(x) = c^\pm$ are characteristic curves. From (2.4), (2.5) it follows that this is also true when $f^\pm(x)$ is replaced by the right hand sides of (2.4).

2.2. Coordinates near $\partial\Omega$

Introduce coordinates (ρ, θ) near $\partial\Omega$, where $\rho = -\Delta(x) \geq 0$ in Ω , $\theta \in [0, 2\pi]$ is a parameter on $\partial\Omega$. One can extend such coordinates to the whole domain $\Omega \setminus \mathcal{O}$ but we shall only use them when $0 \leq \rho \leq \rho_0$ for some small $\rho_0 > 0$. In (ρ, θ) coordinates, (2.1) is replaced by

$$g^{\rho\rho}\xi_\rho^2 + 2g^{\rho\theta}\xi_\rho\xi_\theta + g^{\theta\theta}\xi_\theta^2 = 0, \quad (\xi_\rho, \xi_\theta) \neq (0, 0).$$

Note that

$$(g^{\rho\theta})^2 - g^{\rho\rho}g^{\theta\theta} = C^2(\rho, \theta)\rho,$$

where $C(\rho, \theta) > 0$. We shall denote $C^2(\rho, \theta)\rho$ by $\rho_1(\rho, \theta)$. Thus $C\sqrt{\rho} = \sqrt{\rho_1}$. Either $g^{\rho\rho}$ or $g^{\theta\theta}$ is not zero when $\rho = 0$ since the rank of $[g^{jk}]_{j,k=1}^2$ is 1 on $\partial\Omega$. Let $\rho = 0, \theta = \theta_0$ be such that $g^{\theta\theta}(0, \theta_0) \neq 0$. Near $(0, \theta_0)$, we write the solutions

$$\xi_\theta^\pm = \frac{-g^{\rho\theta} \pm \sqrt{\rho_1}}{g^{\theta\theta}} \xi_\rho^\pm.$$

Then in (ρ, θ) coordinates, (2.4) gives

$$\begin{aligned} \frac{d\rho^\pm}{dx_0} &= \frac{g^{\rho\rho}\xi_\rho^\pm + g^{\rho\theta}\xi_\theta^\pm}{g^{0\rho}\xi_\rho^\pm + g^{0\theta}\xi_\theta^\pm} = \frac{g^{\rho\rho} + g^{\rho\theta} \frac{-g^{\rho\theta} \pm \sqrt{\rho_1}}{g^{\theta\theta}}}{g^{0\rho} + g^{0\theta} \frac{-g^{\rho\theta} \pm \sqrt{\rho_1}}{g^{\theta\theta}}} = \frac{-\rho_1 \pm g^{\rho\theta} \sqrt{\rho_1}}{b(\rho, \theta) \pm g^{0\theta} \sqrt{\rho_1}} \\ \frac{d\theta^\pm}{dx_0} &= \frac{g^{\rho\theta}\xi_\rho^\pm + g^{\theta\theta}\xi_\theta^\pm}{g^{0\rho}\xi_\rho^\pm + g^{0\theta}\xi_\theta^\pm} = \frac{[g^{\rho\theta} + (-g^{\rho\theta} \pm \sqrt{\rho_1})]}{g^{0\rho} + g^{0\theta} \frac{-g^{\rho\theta} \pm \sqrt{\rho_1}}{g^{\theta\theta}}} = \frac{\pm g^{\theta\theta} \sqrt{\rho_1}}{b(\rho, \theta) \pm g^{0\theta} \sqrt{\rho_1}}, \end{aligned} \quad (2.7)$$

where $b(\rho, \theta) = g^{0\rho}g^{\theta\theta} - g^{0\theta}g^{\rho\theta} \neq 0$ (see (2.6)).

2.3. Types of boundary points.

If $g^{\rho\theta}(0, \theta_0) \neq 0$ then

$$\frac{d\rho^\pm}{d\theta} = \mp \frac{\sqrt{\rho_1}}{g^{\theta\theta}} + \frac{g^{\rho\theta}}{g^{\theta\theta}} \quad (2.8)$$

is not zero near $(0, \theta_0)$, i.e. the curve $\rho = \rho^\pm(\theta)$ is transverse to the boundary $\rho = 0$ near $(0, \theta_0)$. It follows from (2.7) that the trajectories $(\rho^\pm(x_0), \theta^\pm(x_0))$ reach the boundary $\rho = 0$ in finite time. Since

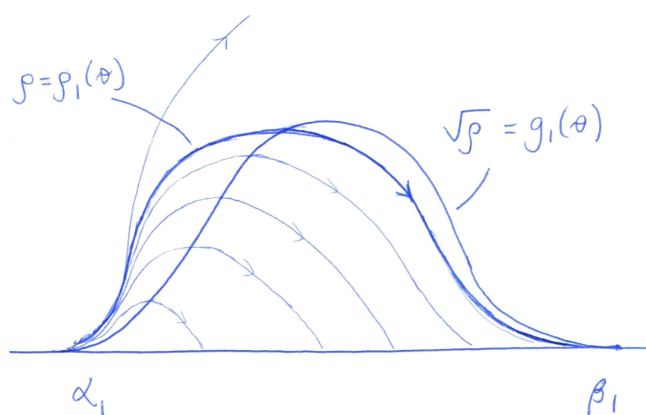


Figure 1. The curve $\rho = \rho_1(\theta)$ is the boundary of all curves of the (+) family starting at α_1 and ending on $(\alpha_1, \beta_1]$.

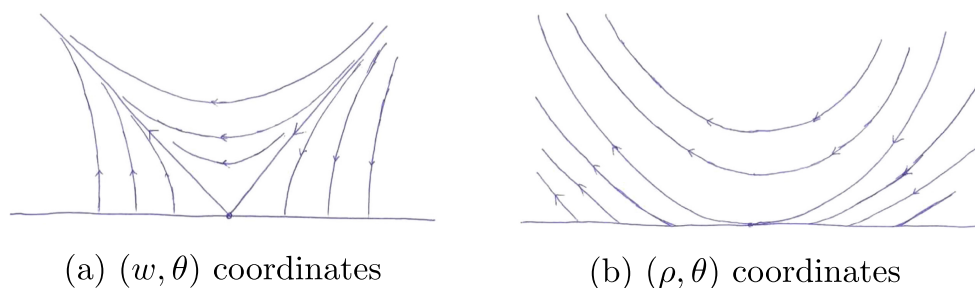


Figure 2. The qualitative picture near a saddle point.

$$\frac{d\rho^\pm}{dx_0} = \pm \frac{g^{\rho\theta}(0, \theta)}{b(0, \theta)} \sqrt{\rho_1} + \mathcal{O}(\rho) \tag{2.9}$$

near $(0, \theta_0)$, one family of trajectories approaches the boundary as x_0 increases while the other leaves the boundary as x_0 increases.

Make a change of variables $w = \sqrt{\rho}$. Denote $w_1 = \sqrt{\rho_1} = Cw$. Since $\frac{d\rho}{dx_0} = 2w \frac{dw}{dx_0}$, so we get

$$2 \frac{dw}{dx_0} = \frac{\pm Cg^{\rho\theta}(w^2, \theta) - C^2w}{b(w^2, \theta) \pm g^{0\theta}(w^2, \theta)w}, \quad \frac{d\theta}{dx_0} = \frac{\pm Cg^{\theta\theta}(w^2, \theta)w}{b(w^2, \theta) \pm g^{0\theta}(w^2, \theta)w}, \tag{2.10}$$

where $b(0, \theta_0) \neq 0, g^{\theta\theta}(0, \theta_0) \neq 0$. If $g^{\rho\theta}(0, \theta_0) = 0$ then $(0, \theta_0)$ is a tangential point of $\partial\Omega$. If $\frac{\partial g^{\rho\theta}}{\partial \theta}(0, \theta_0) \neq 0$ then $(0, \theta_0)$ is a non-degenerate critical point in (w, θ) coordinates. It could be a node, saddle, degenerate node, or spiral restricted to the half-space $w \geq 0$ (cf figures 2–4).

Consider now the case when $g^{\theta\theta}(0, \theta_0) \neq 0, g^{\rho\theta}(0, \theta_0) = 0$, and $g_\theta^{\rho\theta}(0, \theta_0) = 0$. To fix ideas suppose $g^{\theta\theta}(0, \theta_0) < 0$. Then the equation (2.8) has the following form in (w, θ) coordinates:

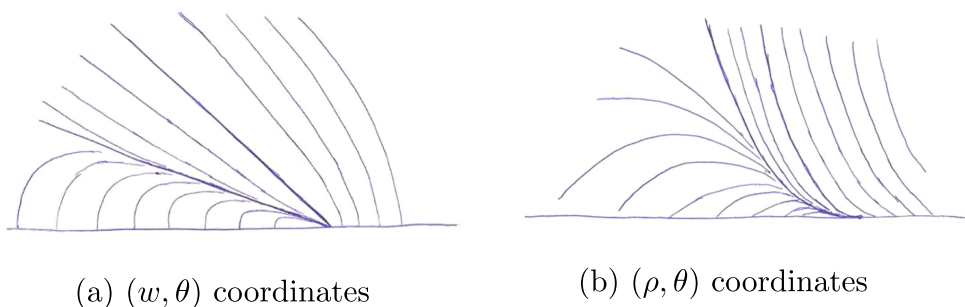


Figure 3. The qualitative picture near a node.

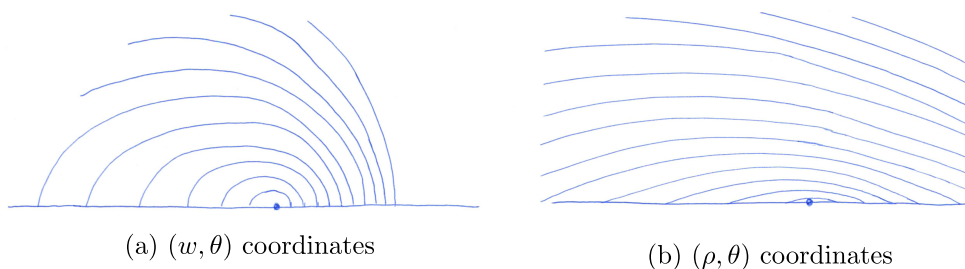


Figure 4. The qualitative picture near a spiral.

$$2w \frac{dw^\pm}{d\theta} = \frac{\mp Cw + g^{\rho\theta}(w^2, \theta)}{g^{\theta\theta}(w^2, \theta)} \tag{2.11}$$

Lemma 2.1. *There is a (+) solution of (2.11) satisfying*

$$w_*^+(\theta) = a_1(\theta) + w_1^+(\theta), \quad |w_1^+(\theta)| \leq C|\theta - \theta_0|^2,$$

defined on $(\theta_0, \theta_0 + \delta)$, $\delta > 0$ small, where

$$a_1(\theta) = \int_{\theta_0}^{\theta} a(\theta') d\theta', \quad a(\theta) = -\frac{1}{2g^{\theta\theta}(0, \theta)}.$$

Analogously, there is a (-) solution of (2.11) satisfying

$$w_*^-(\theta) = -a_1(\theta) + w_1^-(\theta), \quad |w_1^-(\theta)| \leq C|\theta - \theta_0|^2,$$

defined on $(\theta_0 - \delta, \theta_0)$, δ small, with $a_1(\theta)$ as above.

Proof. We rewrite equation (2.11)

$$\frac{dw^+(\theta)}{d\theta} = a(\theta) + \frac{g^{\rho\theta}(0, \theta)}{2wg^{\theta\theta}(0, \theta)} + \frac{g_1(w^2, \theta)}{w},$$

where $|g_1(w^2, \theta)| \leq Cw^2$. Let $g_2(w, \theta) = \frac{g_1(w^2, \theta)}{w}$, $g_3(\theta) = \frac{g^{\rho\theta}(0, \theta)}{2g^{\theta\theta}(0, \theta)}$. Then for $w_1^+(\theta)$ we get

$$\frac{dw_1^+}{d\theta} = \frac{g_3(\theta)}{a_1(\theta) + w_1^+(\theta)} + g_2(a_1(\theta) + w_1^+(\theta)), \quad w_1^+(\theta_0) = 0. \quad (2.12)$$

Let B be the Banach space with norm $\|h\| = \sup_{\theta_0 \leq \theta \leq \theta_0 + \delta} \frac{|h(\theta)|}{(\theta - \theta_0)^2}$. The integral from θ_0 to θ of the right hand side of (2.12) is a contraction mapping in B if δ is small. Therefore $w_1^+(\theta)$ exists.

The proof for $w_1^-(\theta)$ is similar. □

Remark 2.2. Note that the lemma remains valid when $g^{\rho\theta}(0, \theta_0) \neq 0$ but is small. ◇

Remark 2.3. Suppose $\partial\Omega$ contains a characteristic segment L . Let \hat{x} be an interior point of L . Since the boundary $w = 0$ is characteristic for all θ in a neighborhood of \hat{x} we have $g^{\rho\theta}(0, \theta) = 0$, i.e. $g^{\rho\theta}(w^2, \theta) = \mathcal{O}(w^2)$. Therefore by (2.11),

$$\frac{dw^\pm}{d\theta} = \frac{\mp C}{2g^{\theta\theta}(w^2, \theta)} + \mathcal{O}(w).$$

Note that $g^{\theta\theta}(w^2, \theta) \neq 0$ near \hat{x} . Also,

$$\frac{dw^\pm}{dx_0} = \frac{-wC^2}{2b(w^2, \theta)} + \mathcal{O}(w^2).$$

Since $b(w^2, \theta) < 0$ we have that $w^\pm(x_0)$ increases when x_0 increases. Therefore we have two zero-energy null-geodesics on the set $w \geq 0$ that start at \hat{x} .

In (ρ, θ) coordinates these two zero-energy null-geodesics are tangent to the boundary $\rho = 0$. The same picture is true for any θ_1 close to θ_0 . Note that $w = 0$ is an envelope of both the (+) and (-) families near $(0, \theta_0)$. Also note that the point $(0, \theta_0)$ where $g^{\theta\theta}(0, \theta_0) = 0$ is not characteristic. □

3. Existence of a black hole

We shall consider the case when $b_1(\theta) < 0$ and show the existence of a black hole. The case when $b_1(\theta) > 0$ may be treated similarly.

Consider a small circle $\{|x| = \varepsilon\}$ around O . Since $b_1 < 0$, an integral curve of either the (+) or (-) family starting at $\{|x| = \varepsilon\}$ goes to O as x_0 increases, i.e. $\{|x| < \varepsilon\}$ is a trapped region; see [Esk10]. Let Ω^+ be the union of all trajectories of the (+) family in $\Omega \setminus \{|x| \leq \varepsilon\}$ which end at $\{|x| = \varepsilon\}$, i.e.

$$\begin{aligned} \Omega^+ &= \{x^+(x_0) \mid x_0 \in (\ell, 0) \text{ where } -\infty \leq \ell < 0; x^+ \text{ solves (2.4);} \\ &x^+(x_0) \in \Omega \text{ for } x_0 \in (\ell, 0); x^+(0) \in \{|x| = \varepsilon\}; \text{ and } x^+(\ell) \in \partial\Omega \text{ when } \ell > -\infty\}. \end{aligned} \quad (3.1)$$

Lemma 3.1. *Suppose $z_0 \in \partial\Omega^+$ is an interior point of Ω . Let γ_0^+ be a curve of the (+) family passing through z_0 , parameterized $x = x^+(x_0)$. Then there are two possibilities:*

1. γ_0^+ is a characteristic segment with endpoints $\alpha_1, \alpha_2 \in \partial\Omega$, with γ_0^+ tangent to $\partial\Omega$ at $\alpha_1 = \lim_{x_0 \rightarrow +\infty} x^+(x_0)$.
2. γ_0^+ is a smooth closed periodic orbit. In both cases, $\gamma_0^+ \subseteq \partial\Omega^+$.

Proof. Note that $z_0 \notin \Omega^+$ since Ω^+ is open. First suppose the curve γ_0^+ has endpoints $\alpha_1, \alpha_2 \in \partial\Omega$ with $x^+(x_0)$ directed toward α_1 when x_0 increases. Since z_0 is an interior point of Ω , there is a small neighborhood U_ε of z_0 contained in Ω . The curves of the (+) family passing through points of U_ε form a ‘strip’ V_ε .

Since $z_0 \in \partial\Omega^+$ there exist $z_n, z'_n \in U_\varepsilon, z_n \rightarrow z_0, z'_n \rightarrow z_0$ such that $z_n \in \Omega^+, z'_n \notin \Omega^+$. Therefore there are (+) trajectories $x_n(x_0)$ in the strip V_ε belonging to Ω^+ with $x_n(x_{0n}) = z_n$, and trajectories $x'_n(x_0)$ not belonging to Ω^+ with $x'_n(x_{0n}') = z'_n$. If $z^{(1)}$ is any other interior point of γ_0^+ then the trajectories $x_n(x_0)$ and $x'_n(x_0)$ come arbitrarily close to $z^{(1)}$. Therefore $z^{(1)} \in \partial\Omega^+$.

We claim γ_0^+ is tangent to $\partial\Omega$ at α_1 . Indeed if γ_0^+ is transversal to $\partial\Omega$ at α_1 , then all (+) curves in V_ε also intersect $\partial\Omega$ transversally when $\varepsilon > 0$ is small. Therefore all (+) curves in V_ε end at $\partial\Omega$, and do not reach $\{|x| = \varepsilon\}$. This contradicts the fact that V_ε contains (+) curves belonging to Ω^+ .

To show $\alpha_1 = \lim_{x_0 \rightarrow +\infty} x^+(x_0)$, we use (2.10). Since $g^{\rho\theta}(0, \theta_0) = 0$ we have $|g^{\rho\theta}(w^2, \theta)| \leq C(w + |\theta - \theta_0|)$. Thus,

$$\left| \frac{dw}{dx_0} \right| \leq C(w + |\theta - \theta_0|), \quad \left| \frac{d(\theta - \theta_0)}{dx_0} \right| \leq Cw.$$

Therefore

$$\left| \frac{d(w + |\theta - \theta_0|)}{dx_0} \right| \leq C(w + |\theta - \theta_0|),$$

and

$$|dx_0| \geq \frac{1}{C} \frac{d(w + |\theta - \theta_0|)}{w + |\theta - \theta_0|}.$$

Hence $x_0 \rightarrow +\infty$ when $w + |\theta - \theta_0| \rightarrow 0$. At the point α_2 the curve γ_0^+ may be either transversal or tangent to $\partial\Omega$. If it is tangent then, analogously, $\alpha_2 = \lim_{x_0 \rightarrow -\infty} x^+(x_0)$.

If γ_0^+ can be extended indefinitely when $x_0 \rightarrow +\infty$ or $-\infty$ without approaching $\partial\Omega$ then the corresponding limit set of γ_0^+ is a closed orbit γ_1^+ by the Poincaré–Bendixson theorem. Since $\gamma_0^+ \subseteq \partial\Omega^+$, we also have $\gamma_1^+ \subseteq \partial\Omega^+$, and hence $\gamma_0^+ = \gamma_1^+ = \partial\Omega^+$. This concludes the proof of lemma 3.1. \square

3.1. The case of finitely many tangential points

3.1.1. Construction of the event horizon. Suppose the ergosphere $\partial\Omega$ has a finite number of points $\alpha_1, \dots, \alpha_m$ such that the normals to $\partial\Omega$ at $\alpha_p, 1 \leq p \leq m$ are characteristic directions, i.e. $\sum_{j,k=1}^2 g^{jk}(\alpha_p) \nu_j(\alpha_p) \nu_k(\alpha_p) = 0$ where $\nu = (\nu_1, \nu_2)$ is the outward normal to $\partial\Omega$. In other words, the vector fields f^\pm are tangent to $\partial\Omega$ at $x = \alpha_p, 1 \leq p \leq m$.

As in lemma 3.1, let $z_0 \in \partial\Omega^+$ be an interior point of Ω , and let $\gamma_0 \subseteq \partial\Omega^+$ be a characteristic curve passing through z_0 . Suppose that γ_0 can be continued indefinitely as x_0 decreases and does not approach $\partial\Omega$, and hence γ_0 is a closed periodic orbit belonging to the (+) family. If the trajectories of the (−) family are directed inside γ_0 when x_0 increases then $\mathbb{R} \times \gamma_0$ is a black hole event horizon, while if they are directed outside it is a white hole event horizon. In the latter case any trajectory of the (−) family ending at $S_\varepsilon = \{|x| = \varepsilon\}$ cannot reach γ_0 as x_0 decreases. Therefore by the Poincaré–Bendixson theorem there exists a periodic orbit γ_1 inside the domain bounded by γ_0 and belonging to the (−) family. Then $\mathbb{R} \times \gamma_1$ is a black hole event horizon.

Suppose that instead γ_0 is a characteristic segment connecting points $\beta_{02}, \beta_{01} \in \partial\Omega$, where by lemma 3.1 at least one of β_{0j} must be a characteristic point. Then γ_0 divides the domain Ω into parts Ω_1, Ω'_1 and we shall assume that Ω_1 contains Ω^+ . Then consider Ω_1 instead of Ω . Suppose $z_1 \in \partial\Omega^+$ is in the interior of Ω_1 , and let γ_1 be a characteristic segment with endpoints $\beta_{11}, \beta_{12} \in \partial\Omega_1$, where at least one of β_{11}, β_{12} is a tangential point.

Since γ_0 and γ_1 belong to the (+) family it is impossible that an endpoint of γ_1 belongs to the interior of γ_0 , and similarly an endpoint of γ_0 does not coincide with an endpoint of γ_1 unless both curves are tangential to $\partial\Omega$ at this point. Thus the only possibilities are that γ_0, γ_1 do not intersect, or they share a common tangential point in $\partial\Omega$. The curve γ_1 divides Ω_1 into pieces Ω_2, Ω'_2 , where Ω_2 contains Ω^+ . Replace Ω_1 by Ω_2 .

If there is a point $z_2 \in \partial\Omega^+$ such that z_2 is an interior point of Ω_2 , we repeat the previous argument, etc. After finitely many steps we get a domain Ω_r such that $\overline{\Omega^+} = \overline{\Omega_r}$ and Ω_r is a domain whose boundary consists of a finite number of characteristic segments $\gamma_0, \gamma_1, \dots, \gamma_{r-1}$ of the (+) family and a finite number of segments $\delta_1, \dots, \delta_q$ of $\partial\Omega$. Since $\delta_j \subseteq \partial\Omega \cap \overline{\Omega^+}$, $1 \leq j \leq q$, the (+) family of solutions must be directed into Ω^+ on $\bigcup_{j=1}^q \delta_j$.

Denote by Ω_r^- the union of all trajectories of the (−) family in Ω_r that end on S_ε . Note that Ω_r^- is an open set.

Since the open segments $\delta_j, 1 \leq j \leq q$, are not characteristic, and trajectories of the (+) family start on δ_j , we have that trajectories of the (−) family end on $\delta_j, 1 \leq j \leq q$. Note that no trajectory of the (−) family that ends on S_ε can end on $\delta_j, 1 \leq j \leq q$. This means that $\overline{\Omega_r^-}$ does not touch the interior of $\delta_j, 1 \leq j \leq q$.

Now apply to the trajectories of (−) family in Ω_r and Ω_r^- the same arguments as for the trajectories of (+) family in Ω and Ω^+ .

After a finite number of steps we get a domain Ω_{r+p} such that $\overline{\Omega_{r+p}} = \overline{\Omega_r^-}$ and the boundary of Ω_{r+p} consists of a finite number of characteristic segments $\gamma_0^-, \dots, \gamma_{p-1}^-$ of the (−) family and some of the characteristic segments $\gamma_0, \dots, \gamma_{r-1}$, or parts of them, belonging to the (+) family. Since some of the segments γ_k may have been truncated by the above procedure, the boundary of Ω_{r+p} may not be smooth, as some γ_j^-, γ_k may intersect at a corner (cf section 4).

3.1.2. The domain $\mathbb{R} \times \Omega_{r+p}$ is a black hole. To show that $\mathbb{R} \times \Omega_{r+p}$ is a black hole we shall show that any point $(\hat{x}_0, \hat{x}) \in \mathbb{R} \times \partial\Omega_{r+p}$ is a no-escape point. More precisely, let $K_+(\hat{x})$ be the forward light cone at (\hat{x}_0, \hat{x}) , consisting of all $(\hat{x}_0, \hat{x}_1, \hat{x}_2) \in \mathbb{R}^1 \times \mathbb{R}^2$ such that $\sum_{j,k=0}^2 g_{jk}(\hat{x}) \hat{x}_j \hat{x}_k > 0, \hat{x}_0 > 0$. Denote by $\Pi_{\nu(\hat{x})}^\pm$ the half-space $\{(\alpha_0, \alpha_1, \alpha_2) \mid \alpha_1 \nu_1 + \alpha_2 \nu_2 \gtrless 0\}$, where $\nu(\hat{x}) = (\nu_1, \nu_2)$ is the outward normal to $\partial\Omega_{r+p}$ at $x = \hat{x}$. Then \hat{x} is a point of no escape if $K_+(\hat{x}) \subseteq \Pi_{\nu(\hat{x})}^-$ for all $x_0 \in \mathbb{R}$, see [Esk10]. We have several cases to consider.

Let \hat{x} be an interior point of the characteristic segment $\gamma \subseteq \partial\Omega$. If γ belongs to the (+) family then the construction of Ω_{r+p} shows that the curves of the (−) family intersect γ and

directed inside Ω_{r+p} when x_0 increases. Since γ is a characteristic curve it follows from [Esk10] that $K_+(\hat{x})$ is contained in either Π_ν^+ or in Π_ν^- . The tangent vector of the curve of the $(-)$ family passing through \hat{x} is the projection onto (x_1, x_2) of a forward null bicharacteristic and belongs to Π_ν^- . Therefore $K_+(\hat{x}) \subseteq \Pi_\nu^-$, i.e. \hat{x} is a point of no escape.

If γ is a characteristic curve of the $(-)$ family and $\tilde{x} \in \gamma$, then the $(+)$ family curve passing through \tilde{x} is the projection of a forward null bicharacteristic and its tangent vector at \tilde{x} belongs to $\Pi_{\nu_1}^-$, so again $K_+(\tilde{x}) \subseteq \Pi_{\nu_1}^-$, i.e. \tilde{x} is also a point of no escape. Here ν_1 is the exterior normal to $\gamma^{(1)}$ at \tilde{x} . Let $x^{(1)}$ be a corner point of $\partial\Omega_{r+p}$ at the intersection of characteristic segments γ_+, γ_- belonging to the $(+), (-)$ families respectively. Let ν_+, ν_- be the exterior normals to γ_+, γ_- at the point $x^{(1)}$. As above we get that $K_+(x^{(1)}) \subseteq \Pi_{\nu_+}^-, K_+(x^{(1)}) \subseteq \Pi_{\nu_-}^-$, i.e. $K_+(x^{(1)}) \subseteq \Pi_{\nu_+}^- \cap \Pi_{\nu_-}^-$. Therefore $x^{(1)} \in \partial\Omega_{r+p}$ is also a point of no escape, since any vector of $K_+(x^{(1)})$ points inside Ω_{r+p} .

Let now $\hat{x} \in \partial\Omega_{r+p}$ be a tangential point on $\partial\Omega$. It follows from [Esk10] that either $K_+(\hat{x}) \subseteq \Pi_\nu^+$ or $K_+(\hat{x}) \subseteq \Pi_\nu^-$. Since $\partial\Omega$ is the ergosphere, $g_{00}(\hat{x}) = 0$ [Esk10]. Thus $(\dot{x}_0, \dot{x}_1, \dot{x}_2) = (1, 0, 0)$ belongs to $\overline{K_+(\hat{x})}$ since $\sum_{j,k=0}^2 g_{jk}(\hat{x})\dot{x}_j\dot{x}_k = g_{00}(\hat{x}) = 0$. Therefore $\overline{K_+(\hat{x})}$ is tangent to the plane $\dot{x}_1\nu_1 + \dot{x}_2\nu_2 = 0$. Here $\nu = (\nu_1, \nu_2)$ is the outward normal to $\partial\Omega_{r+p}$ at \hat{x} .

Suppose for a moment that $K_+(\hat{x}) \subseteq \Pi_\nu^-$. Since $(1, 0, 0) \in \overline{K_+(\hat{x})}$, for any small $\varepsilon > 0$, $(1, \varepsilon\dot{x}_1, \varepsilon\dot{x}_2) \in K_+(\hat{x})$ when $\dot{x}_1\nu_1 + \dot{x}_2\nu_2 < 0$, for arbitrary (\dot{x}_1, \dot{x}_2) . Therefore $K_+(\hat{x}) = \Pi_\nu^-$ when \hat{x} is a tangential point. Similarly, if $K_+(\hat{x}) \subseteq \Pi_\nu^+$, then $K_+(\hat{x}) = \Pi_\nu^+$.

Let $\hat{x}_n \rightarrow \hat{x}$, where \hat{x} is a tangential point in $\partial\Omega$, and each $\hat{x}_n \in \partial\Omega_{r+p}$ is an interior point of Ω . The points \hat{x}_n are no-escape points for $\partial\Omega_{r+p}$, as was proven above. Note that $\partial\Omega_{r+p}$ is smooth in a neighborhood of \hat{x} . Since \hat{x}_n are no-escape points we have $K_+(\hat{x}_n) \subseteq \Pi_{\nu_n}^-$, where ν_n is the outward unit normal to $\partial\Omega_{r+p}$ at \hat{x}_n . We have $\Pi_{\nu_n}^- \rightarrow \Pi_\nu^-, K_+(\hat{x}_n) \rightarrow K_+(\hat{x})$. Therefore $\overline{K_+(\hat{x})} \subseteq \overline{\Pi_\nu^-}$, i.e. \hat{x} is a point of no escape.

Remark 3.2. These arguments hold for any characteristic point $\hat{x} \in \partial\Omega$ such that there exists a sequence $\hat{x}_n \rightarrow \hat{x}$ with $K_+(\hat{x}_n) \subseteq \Pi_{\nu_n}^-$.

Suppose we have a characteristic segment $\subseteq \partial\Omega$. At the endpoints of the segment we have a sequence of points \hat{x}_n as above. Thus the endpoints are no escape points. For any interior point of the segment we get that $K_+(\hat{x}) \subseteq \Pi_\nu^-$ by continuity. \diamond

Remark 3.3. Note that if $\hat{x} \in \partial\Omega$ is not tangential then it is an escape point: There exists a characteristic direction ν_0 which is not normal to $\partial\Omega$. Since $g_{00}(\hat{x}) = 0$ we have that $K_+(\hat{x})$ is either equal to $\Pi_{\nu_0}^-$ or to $\Pi_{\nu_0}^+$. In both cases there are directions of $K_+(\hat{x})$ which point toward the exterior of Ω . \diamond

Therefore we have proven:

Lemma 3.4. $\mathbb{R} \times \partial\Omega_{r+p}$ is a black hole event horizon.

3.2. The case when $\partial\Omega$ has finitely many characteristic segments and finitely many characteristic points

Suppose there are finitely many open intervals L_1, \dots, L_m in $\partial\Omega$, with $\overline{L_j} \cap \overline{L_k} = \emptyset, j \neq k$, such that the vector fields $f^\pm(x)$ are tangent to $\partial\Omega$ along $L_j, 1 \leq j \leq m$. (Note that $f^+ = f^-$ on $\partial\Omega$.) Assume in addition that there are finitely many isolated tangent points β_1, \dots, β_r .

We again let the open set Ω^+ be as in lemma 3.1, $z_0 \in \partial\Omega^+$ an interior point of Ω , and γ_0 a curve of the (+) family passing through z_0 , with endpoints $\alpha_1, \alpha_2 \in \partial\Omega$ (unless γ_0 is a closed orbit, in which case we are done), (cf the second part of lemma 3.1).

We claim that it is impossible to have $\alpha_1 \in L_{j_1}, \alpha_2 \in L_{j_2}$. If this is the case, consider neighborhoods $U(\alpha_1, \varepsilon_1) \subseteq L_{j_1}, U(\alpha_2, \varepsilon_2) \subseteq L_{j_2}$. For $\varepsilon_1, \varepsilon_2$ small there are solutions of the (+) family $x_\alpha^+(x_0)$ that are close to γ_0 and have endpoints $\alpha \in U(\alpha_1, \varepsilon_1), \tilde{\alpha} \in U(\alpha_2, \varepsilon_2)$. Note that L_j is an envelope for the (+) family (see remark 2.3). All such solutions $x_\alpha^+(x_0)$ are not in Ω^+ , so $z_0 \notin \partial\Omega^+$.

Also, from the proof of lemma 3.1, it is impossible to have $\gamma_0 \subseteq \partial\Omega^+$ which intersects $\partial\Omega$ transversally at both endpoints. Analogously, there is no $\gamma_0 \subseteq \partial\Omega^+$ with one endpoint belonging to some L_j and the other intersecting $\partial\Omega$ transversally.

Therefore γ_0 must have at least one endpoint either among β_1, \dots, β_r or among the endpoints of $\overline{L_1}, \dots, \overline{L_m}$. Thus there are a finite number of such curves. Following the proof of lemma 3.1 we get that the boundary of Ω^+ consists of a finite number of characteristic segments inside Ω of the (+) family, a finite number of the segments $\overline{L_j}, 1 \leq j \leq m$ or closed subintervals of $\overline{L_j}$ and a finite number of segments of $\partial\Omega$ where (+) family trajectories start as x_0 increases. Starting with $\overline{\Omega^+}$ instead of $\overline{\Omega}$ we consider the open set $\Omega_1^- \subseteq \Omega^+$ of (-) family trajectories ending on S_ε , and it is clear that we may repeat the proof of lemma 3.1. We get after a finite number of steps that the boundary Ω_1^- consists of a finite number of characteristic segments or parts of characteristic segments inside Ω , some belonging to the (+) family and some to the (-) family and some characteristic segments that are parts of $\bigcup_{j=1}^m L_j$. It follows from the proof of lemma 3.4 and remark 3.2 that $\mathbb{R} \times \Omega^-$ is a black hole event horizon. Note that the boundary of $\partial\Omega_1^-$ may have corners—i.e. it may only be piecewise smooth.

3.3. The general case

Consider Ω^+ . We have that $\overline{\Omega^+}$ does not intersect any of the open intervals (α_k, β_k) in $\partial\Omega$ where (+) family curves end as x_0 increases. There can be at most countably many such intervals. Denote by Ω_k^+ the union of all (+) family curves ending on (α_k, β_k) as x_0 increases. Note that $\Omega_k^+ \cap \Omega^+ = \emptyset$. Take any $z_0 \in \partial\Omega_k^+$ which is an interior point of Ω . Denote by γ_0 the (+) family curve passing through z_0 . Then γ_0 ends at either α_k or β_k , say α_k to fix ideas. Let α_{k_1} be a point on $\partial\Omega$ where γ_0 starts. Denote by Ω_{k_1} the domain bounded by γ_0 and $\partial\Omega$ and not containing O . If Ω_{k_1} contains Ω_k^+ we replace Ω by $\Omega_1 = \Omega \setminus \overline{\Omega_{k_1}}$. If Ω_{k_1} does not contain Ω_k^+ then there is another characteristic curve $\gamma^{(0)}$ belonging to the boundary of Ω_k^+ and ending at β_k . Let $\beta_{k_1} \in \partial\Omega$ be the starting point of $\gamma^{(0)}$. Let $\Omega_k^{(1)}$ be the domain bounded by $\gamma^{(0)}$ and $\partial\Omega$ that contains Ω_{k_1} and Ω_k^+ and we shall replace Ω by $\Omega \setminus \overline{\Omega_k^{(1)}}$. Note that $\partial(\Omega \setminus \overline{\Omega_k^{(1)}})$ does not contain (α_k, β_k) . Note also that $\partial(\Omega \setminus \overline{\Omega_k^{(1)}})$ is smooth at β_k but may have a corner at β_{k_1} . In the latter case β_{k_1} belongs to an open interval (σ, δ) where the curves of the (+) family start. Consider any other interval $(\alpha_j, \beta_j), j \neq k$, where curves of the (+) family end. Let $\Omega_j^{(1)}$ be a domain constructed as with $\Omega_k^{(1)}$. Since curves of the (+) family do not intersect in Ω we have that $\Omega_j^{(1)} \cap \Omega_k^{(1)} = \emptyset$. Note that $\overline{\Omega_j^{(1)}} \cap \overline{\Omega_k^{(1)}}$ is either empty or consists of at most two tangential points in $\partial\Omega$. Denote $\Omega_\infty^+ = \bigcap_{k=1}^\infty (\Omega \setminus \overline{\Omega_j^{(1)}}) = \Omega \setminus \bigcup_{j=1}^\infty \overline{\Omega_j^{(1)}}$.

The boundary $\partial\Omega_\infty^+$ consists of characteristic segments of the (+) family, a closed set of tangent points belonging to $\partial\Omega$, and intervals $(\sigma_k, \delta_k), k = 1, 2, \dots$, or parts of such intervals, where the (+) family of curves start when x_0 increases. We shall show (cf below) that $\partial\Omega_\infty^+$ is smooth, except possibly at a countable number of corner points β_{k_j} belonging to some of the open intervals (σ_k, δ_k) . Now consider the union Ω_k^- of all (-) family curves in Ω that end on

(σ_k, δ_k) when x_0 increases. Let $z_1 \in \partial\Omega_k^-$ be an interior point of Ω_∞^+ and let γ_1^- be the $(-)$ family curve passing through z_1 . Let (σ_k, σ_k) be the endpoints of γ_1^- . Consider also the $(-)$ family curve $\gamma_1^{(1)}$ ending at δ_k and belonging to $\partial\Omega_k^-$. Here, it is possible that $\gamma_1^{(1)}$ is a single point. Let $\Omega_k^{(2)}$ be the domain bounded by $\partial\Omega$ and either γ_1^- or $\gamma_1^{(1)}$, which contains Ω_k^- and does not contain \mathcal{O} . To fix ideas let $\gamma_1^- \subseteq \partial\Omega_k^{(1)}$. Then we replace Ω_∞^+ by $\Omega_\infty^+ \setminus \overline{\Omega_k^{(2)}}$.

If we have $\beta_{k_j} \in (\sigma_k, \delta_k) \cap \partial\Omega_\infty^+$ then $\beta_{k_j} \notin \partial(\Omega_\infty^+ \setminus \Omega_k^{(2)})$ since $(\sigma_k, \delta_k) \subseteq \Omega_k^{(2)}$. Denote by $\gamma_1^{(1)}$ the intersection of γ_1^- with $\partial\Omega_\infty^+$. Then the endpoints of $\gamma_1^{(1)}$ are either tangential points of $\partial\Omega$ or corner points of $\partial\Omega_\infty^+$ belonging to the interior of Ω .

Repeating this procedure for all (σ_k, δ_k) , $k = 1, 2, \dots$, and for all characteristic segments γ_j^+ such that the $(-)$ family curves end on γ_j^+ , we get a domain $\Omega_\infty^- \subseteq \Omega_\infty^+$ such that the boundary of Ω_∞^- consists of characteristic segments belonging to either the $(+)$ or $(-)$ family and a closed set $S_1 \subseteq \partial\Omega \cap \partial\Omega_\infty^-$ of tangential points.

We shall show that $\partial\Omega_\infty^-$ is continuously differentiable except at corner points. It is enough to show that $\partial\Omega_\infty^-$ is continuously differentiable at any point of $S_1 = \partial\Omega_\infty^- \cap \partial\Omega$. Let $x^{(0)}$ be any point of S_1 . Introduce (ρ, θ) coordinates in a small neighborhood U_0 of $x^{(0)} = (0, \theta_0)$. We have by (2.7), (2.8),

$$\begin{aligned} \frac{d\rho^\pm}{d\theta} &= \frac{\pm\sqrt{\rho_1} + g^{\rho\theta}(\rho, \theta)}{g^{\theta\theta}(\rho, \theta)}, \\ \frac{dx_0}{d\rho} &= \frac{\pm g^{\rho\theta}(\rho, \theta)\sqrt{\rho_1} - \rho_1}{b(\rho, \theta) \pm g^{\theta\theta}(\rho, \theta)\sqrt{\rho_1}}, \end{aligned} \tag{3.2}$$

where $g^{\theta\theta}(0, \theta_0) < 0$, $b(0, \theta_0) < 0$, $g^{\rho\theta}(0, \theta_0) = 0$. Since U_0 is small we may assume that $g^{\theta\theta} < 0$, $b(\rho, \theta) \pm g^{\theta\theta}(\rho, \theta)\sqrt{\rho_1} < 0$ in U_0 . In U_0 , there are at most countably many intervals (α_k, β_k) where $g^{\rho\theta}(0, \theta) > 0$ or $g^{\rho\theta} < 0$. Let (α_1, β_1) be such that $g^{\rho\theta}(0, \theta) > 0$ on (α_1, β_1) . It follows from (3.2) that curves of the $(+)$ family end on $\{\rho = 0\}$ when x_0 increases. We shall prove that there exists a curve $\rho = \rho_1(\theta)$ of the $(+)$ family starting at α_1 and ending at β_1 such that $\rho = \rho_1(\theta)$ is the boundary of all curves of the $(+)$ family ending on (α_1, β_1) . Let $w = \sqrt{\rho}$. We have $g^{\rho\theta}(w^2, \theta) = c_1(w^2, \theta)w^2 + g^{\rho\theta}(0, \theta)$. Since U_0 is small we have by the contraction mapping theorem that

$$-\sqrt{\rho_1} + g^{\rho\theta}(\rho, \theta) = c_2(\sqrt{\rho}, \theta)(-\sqrt{\rho} + g_1(\theta)),$$

where $c_2 > 0$, $g_1(\theta) > 0$, $\theta \in (\alpha_1, \beta_1)$.

Consider the domain V bounded by $w = g_1(\theta)$ and $w = 0$. We have that $\frac{d\rho}{d\theta} = 0$ when $w = g_1(\theta)$, $\frac{d\rho^+}{d\theta} < 0$ inside V (since $g^{\theta\theta} < 0$) and $\frac{d\rho^+}{d\theta} > 0$ outside of V . Therefore curves $\rho = \rho_+(\theta)$ of the $(+)$ family that end at $(0, \theta')$, $\theta' \in (\alpha_1, \beta_1)$ increase when θ decreases until $\rho = \rho_+(\theta)$ intersects $\sqrt{\rho} = g_1(\theta)$. Then $\rho_+(\theta)$ decreases outside V for $\alpha_1 < \theta < \beta_1$ when θ decreases. Note that $\rho = \rho_+(\theta)$ cannot cross the solution $\rho = (w_*^+(\theta))^2$ constructed in 2.1, since they belong to the same family. Therefore $\rho = \rho_+(\theta)$ must end at $\theta = \alpha_1$ (see figure 1).

Analogously if (α_2, β_2) is an interval in $U \cap \{\rho = 0\}$ such that $g^{\rho\theta}(0, \theta) < 0$, then there exists a $(-)$ family curve $\rho = \rho_2(\theta)$ that starts on β_2 and ends on α_2 such that $\rho = \rho_2(\theta)$ is the boundary for all $(-)$ family curves that end at $(0, \theta)$, where $\theta \in (\alpha_2, \beta_2)$.

Let $\rho = \rho(\theta)$ be a function on $U \cap \{\rho = 0\}$ equal to $\rho_k(\theta)$ on (α_k, β_k) and zero otherwise. The function $\rho = \rho(\theta)$ is the boundary of $\Omega_\infty^- \cap U$.

We shall show that $\rho = \rho(\theta)$ is continuously differentiable at any point $\partial\Omega_\infty^- \cap U$. Let $(0, \theta')$ be any point in U_0 such that $g^{\rho\theta}(0, \theta') = 0$. For any $\varepsilon > 0$ there is $\delta > 0$ such that $|g^{\rho\theta}(0, \theta)| < \varepsilon$ when $|\theta - \theta'| < \delta$. Let (α_j, β_j) be any interval in $(\theta' - \delta, \theta' + \delta)$ such that

$|g^{\rho\theta}(0, \theta)| \neq 0$ on (α_j, β_j) . We have

$$|\rho_j(\theta)| \leq \max_{[\alpha_j, \beta_j]} |g_j(\theta)| \leq C \max_{[\alpha_j, \beta_j]} |g^{\rho\theta}(0, \theta)| < C\varepsilon.$$

Therefore $|\rho(\theta)| < C\varepsilon$ for $(\theta' - \delta, \theta' + \delta)$, i.e. $\lim_{\theta \rightarrow \theta'} \rho(\theta) = 0$. This proves the continuity of $\rho(\theta)$. Analogously, $\left| \frac{d\rho(\theta)}{d\theta} \right| \leq C |g^{\rho\theta}(\rho(\theta), \theta)| + \sqrt{\rho} \leq C(|g^{\rho\theta}(0, \theta)| + \sqrt{\rho})$.

Thus $\lim_{\theta \rightarrow \theta'} \frac{d\rho(\theta)}{d\theta} = 0$, i.e. $\frac{d\rho(\theta)}{d\theta}$ is also continuous.

As in lemma 3.4 and remark 3.2, we get that any point of $\partial\Omega_\infty^-$ is a no-escape point, i.e. $\mathbb{R} \times \Omega_\infty^-$ is a black hole.

Remark 3.5. The black hole constructed in this subsection may be different from the black holes constructed in the previous subsections, in the case when there is more than one black hole [Esk14]. ◇

Remark 3.6. At tangential points, $\partial\Omega_\infty^-$ is C^1 but not C^2 in general, since there are characteristic curves of different families that have a common tangential point. ◇

4. Acoustic metrics and an example with corners

4.1. Acoustic metrics

We consider acoustic waves in a moving medium. The *acoustic metric* associated to a vector field $v = (v_1, v_2)$ is the (stationary) Lorentzian metric $\frac{\rho}{c}[c^2 dx_0^2 - (dx - v dx_0)^2]$, i.e. the metric g given by

$$g_{00} = \frac{\rho}{c}(c^2 - |v|^2), \quad g_{0j} = g_{j0} = \frac{\rho}{c}v_j, \quad 1 \leq j \leq 2, \quad g_{ij} = -\frac{\rho}{c}\delta_{ij}, \quad 1 \leq i, j \leq 2, \quad (4.1)$$

The inverse of the metric tensor is given by

$$g^{00} = \frac{1}{\rho c}, \quad g^{j0} = g^{0j} = \frac{1}{\rho c}v_j, \quad 1 \leq j \leq 2, \quad g^{jk} = \frac{1}{\rho c}(v_j v_k - c^2 \delta_{jk}), \quad 1 \leq j, k \leq 2.$$

We assume that the flow $v = (v_1, v_2)$ is irrotational, i.e. there exists a potential ψ such that $v = \nabla\psi$, barotropic, i.e. $p = p(\rho)$ where p is the pressure and ρ is the density. Moreover, v and ρ satisfy the continuity equation

$$\rho_t + \nabla \cdot (\rho \nabla \psi) = 0,$$

and the Euler equation, which can be reduced to the form [Vis98]

$$\psi_t + h + \frac{1}{2}(\nabla\psi)^2 + \Phi = 0,$$

where Φ represents external forces and $h(p)$ is the specific enthalpy.

In the case when v and ρ satisfy these requirements, the wave equation (1.1) for a metric of the form (4.1) is a physical model for the propagation of sound waves (see [Vis98]) where $c = \sqrt{\frac{dp}{d\rho}}$ is the speed of sound.

We shall take ρ to be constant. Then p and c are constant as well. Then by continuity equation

$$\Delta\psi = 0$$

i.e. ψ is a harmonic function. Rescaling, we shall assume that $c = 1$. Then the ergoregion is where $1 - |v|^2 < 0$.

Remark 4.1. Other well-known spacetime metrics may be transformed into the form (4.1) after an appropriate choice of coordinates, including the Schwarzschild metric in Painlevé–Gullstrand coordinates [Vis98]. \diamond

Remark 4.2. As was noted in the introduction, acoustic metrics are not the only physical examples of analogue (artificial) black holes. There are models for optical black holes, surface waves, relativistic acoustic waves, Bose–Einstein condensates, and others. See the references in the introduction. \diamond

It will be convenient to write the vector field in polar coordinates as $v = v_r \hat{r} + v_\theta \hat{\theta}$, $v_r = \frac{\partial\psi}{\partial r}$, $v_\theta = \frac{1}{r} \frac{\partial\psi}{\partial\theta}$. In this case the vector field is a solution of the Euler equations. We will specify an explicit choice of ψ in the following subsection.

Let

$$v = \frac{A(r, \theta)}{r} \hat{r} + \frac{B(r, \theta)}{r} \hat{\theta}, \quad A, B \in C^\infty,$$

and let g be the corresponding acoustic metric, which satisfies (1.2)–(1.4). In polar coordinates, the form corresponding to (2.1) is

$$\left(\frac{A^2}{r^2} - 1\right)\xi_r^2 + 2\frac{AB}{r^3}\xi_r\xi_\theta + \left(\frac{B^2}{r^4} - \frac{1}{r^2}\right)\xi_\theta^2 = 0,$$

i.e. $g^{rr} = \frac{A^2}{r^2} - 1$, $g^{r\theta} = g^{\theta r} = \frac{AB}{r^3}$, $g^{\theta\theta} = \frac{B^2}{r^4} - \frac{1}{r^2}$. We find the solutions

$$\xi_\theta^\pm = \frac{-\frac{AB}{r^3} \pm \sqrt{\rho}}{\frac{B^2}{r^4} - \frac{1}{r^2}} \xi_r^\pm.$$

In addition, the acoustic metric satisfies $g^{r0} = \frac{A}{r}$, $g^{\theta 0} = \frac{B}{r^2}$. Therefore the system (2.4) becomes

$$\begin{aligned} \frac{dr^\pm}{dx_0} &= \frac{g^{rr}f_2^\pm - g^{r\theta}f_1^\pm}{g^{r0}f_2^\pm - g^{\theta 0}f_1^\pm} = \frac{\left(\frac{A^2}{r^2} - 1\right)f_2^\pm - \frac{AB}{r^3}f_1^\pm}{\frac{A}{r}f_2^\pm - \frac{B}{r^2}f_1^\pm} \\ \frac{d\theta^\pm}{dx_0} &= \frac{g^{\theta r}f_2^\pm - g^{\theta\theta}f_1^\pm}{g^{r0}f_2^\pm - g^{\theta 0}f_1^\pm} = \frac{\frac{AB}{r^3}f_2^\pm - \left(\frac{B^2}{r^4} - \frac{1}{r^2}\right)f_1^\pm}{\frac{A}{r}f_2^\pm - \frac{B}{r^2}f_1^\pm}. \end{aligned} \quad (4.2)$$

Near the ergosphere $A^2 + B^2 - r^2 = \rho = 0$, we can use

$$f_1^\pm = \frac{AB}{r} \mp \sqrt{\rho}, \quad f_2^\pm = \frac{B^2}{r^2} - 1. \quad (4.3)$$

Remark 4.3. Alternatively, formulas for f^\pm which are valid on all of Ω , up to removable singularities, are

$$f_1^\pm = \frac{(A^2 - r^2)(B \mp r)}{\frac{AB}{r} \pm \sqrt{A^2 + B^2 - r^2}}, \quad f_2^\pm = B \mp r. \quad (4.4)$$

◇

Denote

$$b_0 = \frac{A}{r} \left(\frac{B^2}{r^2} - 1 \right) - \frac{B}{r^2} \left(\frac{AB}{r} \mp \sqrt{\rho} \right) = -\frac{A}{r} \pm \frac{B}{r^2} \sqrt{\rho} \quad (4.5)$$

Note that $b_0 > 0$ near $\rho = 0$ since $A < 0$. Therefore

$$\begin{aligned} \frac{dr^\pm}{dx_0} &= \frac{\left(\frac{A^2}{r^2} - 1 \right) \left(\frac{B^2}{r^2} - 1 \right) - \frac{AB}{r^3} \left(\frac{AB}{r} \mp \sqrt{\rho} \right)}{b_0} = \frac{\pm \left(\frac{AB}{r} \mp \sqrt{\rho} \right) \sqrt{\rho}}{b_1} \\ \frac{d\theta^\pm}{dx_0} &= \frac{\frac{AB}{r^3} \left(\frac{B^2}{r^2} - 1 \right) - \left(\frac{B^2}{r^4} - \frac{1}{r^2} \right) \left(\frac{AB}{r} \mp \sqrt{\rho} \right)}{b_0} = \frac{\pm \left(\frac{B^2}{r^2} - 1 \right) \sqrt{\rho}}{b_1}. \end{aligned} \quad (4.6)$$

where $b_1 = r^2 b_0 = -Ar \pm B\sqrt{\rho}$.

For later, we record that in (ρ, θ) coordinates, we have

$$\begin{aligned} \frac{d\rho^\pm}{dx_0} &= 2(AA_\theta + BB_\theta) \frac{d\theta^\pm}{dx_0} + 2(AA_r + BB_r - 2r) \frac{dr^\pm}{d\theta} \\ &= \pm \frac{2Q\sqrt{\rho}}{b_1} + \frac{2(r - AA_r - BB_r)\rho}{b_1}. \end{aligned} \quad (4.7)$$

where

$$Q = (AA_\theta + BB_\theta) \left(\frac{B^2}{r^2} - 1 \right) + (AA_r + BB_r - r) \frac{AB}{r}. \quad (4.8)$$

Since $b_1 > 0$, and $\frac{B^2}{r^2} - 1 < 0$ near $\rho = 0$, we have that $\frac{d\rho^\pm}{dx_0} \leq 0$, i.e. $\theta^+(x_0)$ decreases and $\theta^-(x_0)$ increases when x_0 increases. We have

$$\frac{d\rho^\pm}{d\theta} = \frac{2Q}{\frac{B^2}{r^2} - 1} \mp \frac{2(AA_r + BB_r - r)\sqrt{\rho}}{\frac{B^2}{r^2} - 1}. \quad (4.9)$$

Therefore $\rho^\pm = \rho^\pm(\theta)$ is tangential to $\rho = 0$ if and only if $Q = 0$.

It follows from (4.7) that near $\rho = 0$, $\frac{d\rho^\pm}{dx_0} < 0$ when $Q < 0$ and $\frac{d\rho^\pm}{dx_0} > 0$ when $Q > 0$. Therefore $(\rho^+(x_0), \theta^+(x_0))$ ends on $\rho = 0$ when $Q < 0$ and $(\rho^+(\theta_0), \theta^+(x_0))$ starts on $\rho = 0$

when $Q > 0$. Similarly $(\rho^-(x_0), \theta^-(x_0))$ starts on $\rho = 0$ when $Q < 0$ and ends on $\rho = 0$ when $Q > 0$.

4.2. Example of an acoustic black hole with a corner

Consider a potential

$$\psi = A_0 \log r + \varepsilon r \sin \theta, \quad A_0 < -1, 0 < \varepsilon < 1,$$

so that

$$A = r \frac{\partial \psi}{\partial r} = A_0 + \varepsilon r \sin \theta, \quad B = \frac{\partial \psi}{\partial \theta} = \varepsilon r \cos \theta.$$

In (w, θ) coordinates, from (4.6), (4.7) we have

$$\begin{aligned} \frac{dw^\pm}{dx_0} &= \frac{\pm Q + (r - (A_0 + \varepsilon r \sin \theta)\varepsilon \sin \theta - (\varepsilon r \cos \theta)\varepsilon \cos \theta)w}{-(A_0 + \varepsilon r \sin \theta)r \pm (\varepsilon r \cos \theta)w} \\ \frac{d\theta^\pm}{dx_0} &= \frac{\pm((\varepsilon \cos \theta)^2 - 1)w}{-(A_0 + \varepsilon r \sin \theta)r \pm (\varepsilon r \cos \theta)w}, \end{aligned} \tag{4.10}$$

where

$$Q = \varepsilon \cos \theta (A_0^2 \varepsilon \sin \theta + r^2 \varepsilon (\varepsilon^2 - 1) \sin \theta + 2A_0 r (\varepsilon^2 - 1)).$$

The equation of the ergosphere $w = 0$ is $(A_0 + \varepsilon r \sin \theta)^2 + (\varepsilon r \cos \theta)^2 - r^2 = 0$, which gives

$$r = r_0(\theta) = \frac{-A_0}{1 - \varepsilon^2} (-\varepsilon \sin \theta + \sqrt{1 - \varepsilon^2 \cos^2 \theta}).$$

Note that $r_0(\pi/2) = \frac{-A_0}{1 + \varepsilon}$, $r_0(-\pi/2) = \frac{-A_0}{1 - \varepsilon}$, and $\frac{-A_0}{1 + \varepsilon} \leq r(\theta) \leq \frac{-A_0}{1 - \varepsilon}$ for all θ .

When $w = 0$, we have $Q = -2A_0(\varepsilon r \cos \theta)(A_0 + \varepsilon r \sin \theta)$. Thus there are tangential points where $w = 0$ and $\theta = \pm \pi/2$. If $w = 0$ and $\theta \neq \pm \pi/2$, we can only have tangential points when $A_0 + \varepsilon r \sin \theta = 0$ and hence $(\varepsilon r \cos \theta)^2 = r^2$, which is impossible when $|\varepsilon| < 1$.

- At the point $w = 0, \theta = \pi/2$, the linearization in (w, θ) has the Jacobian matrix

$$\begin{bmatrix} -\frac{(1 + \varepsilon)^2}{A_0} & \mp 2\varepsilon(1 + \varepsilon) \\ \mp \frac{(1 + \varepsilon)^2}{A_0^2} & 0 \end{bmatrix},$$

which has determinant $-2\varepsilon(1 + \varepsilon)^3/A_0^2 < 0$. Therefore $w = 0, \theta = \pi/2$ is a saddle point.

- At the points $w = 0, \theta = -\pi/2$, the linearization in (w, θ) has the Jacobian matrix

$$\begin{bmatrix} -\frac{(1 - \varepsilon)^2}{A_0} & \pm 2\varepsilon(1 - \varepsilon) \\ \mp \frac{(1 - \varepsilon)^2}{A_0^2} & 0 \end{bmatrix},$$

which has determinant $2\varepsilon(1 - \varepsilon)^3/A_0^2 > 0$, trace $-(1 - \varepsilon)^2/A_0 > 0$, and discriminant $(1 - \varepsilon)^4/A_0^2 - 8\varepsilon(1 - \varepsilon)^3/A_0^2 = (1 - \varepsilon)^3(1 - 9\varepsilon)/A_0^2$. Therefore $w = 0, \theta = -\pi/2$ is an unstable node for $0 < \varepsilon < \frac{1}{9}$ and an unstable spiral for $\frac{1}{9} < \varepsilon < 1$.

In the next subsection we will show that from these calculations we can conclude that the black hole has a corner whenever the second critical point is a spiral.

4.3. Phase portrait with two critical points

In this subsection we describe the generic phase portrait when there are two critical points.

4.3.1. One saddle and one spiral. Consider first the case of one saddle point $\alpha_1 = \{\rho = 0, \theta = \pi/2\}$ and one spiral $\alpha_2 = \{\rho = 0, \theta = -\pi/2\}$. Let us assume, to fix ideas, that that point $\alpha_1 = \{\rho = 0, \theta = \pi/2\}$ is a saddle, the point $\alpha_2 = \{\rho = 0, \theta = -\pi/2\}$ is an unstable spiral and the (+) trajectories end on $\{\rho = 0, 3\pi/2 < \theta < \pi/2\}$ and start on $\{\rho = 0, -\pi/2 < \theta < \pi/2\}$ when x_0 increases. Note that $\theta = -\pi/2 = 3\pi/2 \pmod{2\pi}$ is the same point.

The (+) trajectory γ^+ that ends at α_1 must start at some point $\{\rho = 0, \theta = \theta^+\}$ where $-\pi/2 < \theta^+ < \pi/2$. The (+) trajectories starting on $\{\rho = 0, -\pi/2 < \theta < \theta^+\}$ must end on $\{\rho = 0, \pi/2 < \theta < 3\pi/2\}$. The (+) trajectories starting on $\{\rho = 0, \theta^+ < \theta \leq \pi/2\}$ must approach O when x_0 increases. Therefore the set Ω^+ of all (+) trajectories ending at O is bounded by γ^+ and $\{\rho = 0, \theta^+ \leq \theta \leq \pi/2\}$. Analogously there exists a (-) trajectory γ^- that ends at α_1 and starts at some point $\{\rho = 0, \theta = \theta^-\}$ with $\pi/2 < \theta^- < 3\pi/2$. The set Ω^- of all (-) trajectories ending at O is bounded by γ^- and $\{\rho = 0, \pi/2 \leq \theta \leq \theta^-\}$. Thus the black hole $\Omega_0 = \Omega^+ \cap \Omega^-$ is bounded by segments of γ^+ and γ^- which meet at a corner point. The numerically computed phase portraits in figure 5 for $A = A_0 + \varepsilon r \sin \theta$, $B = \varepsilon r \cos \theta$, with $A_0 = -2.0$ and $\varepsilon = 0.3$, indicate trajectories approximating γ^+ and γ^- as described above. Combining the pictures in figures 5(a) and (b) we get a black hole with a corner. See also figure 6.

4.3.2. One saddle and one node. Now we consider the slightly more difficult case where there is one saddle point and one node. As before we assume that the (+) trajectories end on $\{\rho = 0, \pi/2 < \theta < 3\pi/2\}$ and start on $\{\rho = 0, -\pi/2 < \theta < \pi/2\}$, and let assume that $\alpha_1 = \{\rho = 0, \theta = \pi/2\}$ is a saddle point and $\alpha_2 = \{\rho = 0, \theta = 3\pi/2\}$ is an unstable node.

Consider all (+) trajectories that start at the node α_2 . There are two cases.

In the first case, the endpoints of the (+) trajectories starting at the node cover the interval $\{\rho = 0, \pi/2 < \theta < 3\pi/2\}$ of the ergosphere. It follows that there is a (+) trajectory γ_1^+ starting at the node α_2 and ending at the saddle point α_1 . More precisely, γ_1^+ approaches the node when $x_0 \rightarrow -\infty$ and approaches the saddle when $x_0 \rightarrow +\infty$. There can be (+) trajectories emerging from the node that do not end on $\{\rho = 0, \pi/2 \leq \theta \leq 3\pi/2\}$. These trajectories must all end at the singularity O . Also, all (+) trajectories starting on $\{\rho = 0, -\pi/2 < \theta < \pi/2\}$ end at O . Therefore, the set Ω^+ of all trajectories that end at O is bounded by γ_1^+ and the part of the ergosphere $\{\rho = 0, -\pi/2 \leq \theta \leq \pi/2\}$.

In the second case there exists $\pi/2 < \theta_1^+ < 3\pi/2$ such that the endpoints of the (+) trajectories starting at the node cover the interval $\{\rho = 0, \theta_1^+ \leq \theta < 3\pi/2\}$ of the ergosphere. Therefore there is a (+) trajectory γ_2^+ ending at the saddle point α_2 that starts at some point $\{\rho = 0, \theta = \theta_2^+\}$, where $-\pi/2 < \theta_2^+ < \pi/2$. All (+) trajectories starting on $\{\rho = 0, -\pi/2 < \theta < \theta_2^+\}$ end on $\{\rho = 0, \theta_1^+ < \theta < \pi/2\}$ and all (+) trajectories starting on $\{\rho = 0, \theta_2^+ < \theta < \pi/2\}$ end at the singularity O , including the (+) trajectory starting at $\alpha_2 = \{\rho = 0, \theta = \pi/2\}$. Therefore the set Ω^+ of all (+) trajectories approaching O is bounded by γ_2^+ and the part of the ergosphere $\{\rho = 0, \theta_2^+ < \theta < \pi/2\}$.

For the (-) trajectories there are also two cases. In one case there is a (-) trajectory γ_1^- that starts at some point $\{\rho = 0, \theta = \theta_2^-\}$, where $\pi/2 < \theta_2^- < 3\pi/2$, and ends at the saddle

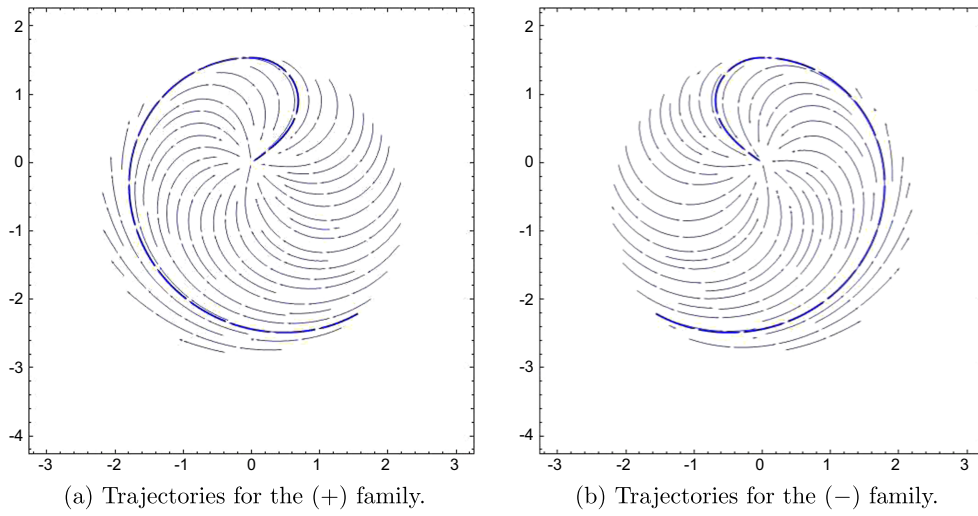


Figure 5. Numerically plotted trajectories for (4.6) with $A = A_0 + \varepsilon r \sin \theta$, $B = \varepsilon r \cos \theta$, $A_0 = -2.0$, $\varepsilon = 0.3$. The bold trajectories pass through $\theta = -\pi/2$, $r = 2.4350096$.

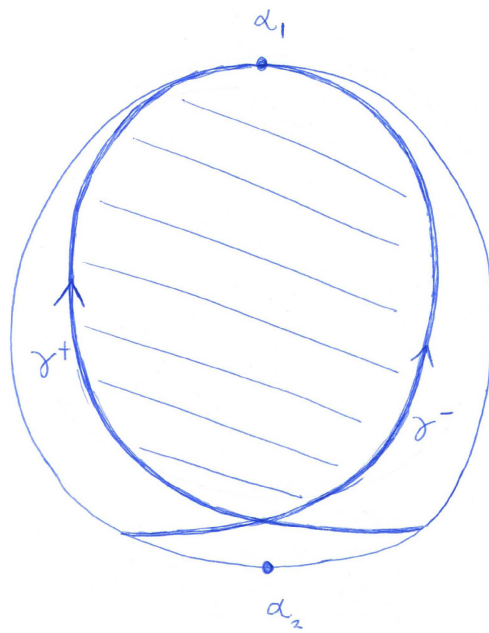


Figure 6. Qualitative sketch of a black hole with a corner in the case of two critical points.

point α_1 . The set Ω^- of $(-)$ trajectories ending at O is bounded by γ_2^- and the part of the ergosphere $\{\rho = 0, \pi/2 < \theta < \theta_2^-\}$. In the other case Ω^- is bounded by a $(-)$ trajectory γ_2^- starting at α_2 and ending at α_1 , and by the part of the ergosphere $\{\rho = 0, \pi/2 < \theta < 3\pi/2\}$. The black hole Ω_0 is the intersection of Ω^+ and Ω^- . Therefore Ω_0 is bounded by (parts of) γ_1^+

or γ_2^+ or γ_1^- or γ_2^- . Only in the case when Ω_0 is bounded by γ_1^+ and γ_1^- is the boundary $\partial\Omega_0$ smooth. In the three other cases $\partial\Omega_0$ has a corner point. We do not present numerical investigations of this case.

As in remark 3.6, we note that even when $\partial\Omega_0$ is smooth it is C^1 but may not be C^2 since $\partial\Omega_0$ consists of two smooth curves γ_1^+ , γ_2^+ tangential to the ergosphere at α_1 and α_2 and belonging to different families.

5. Determination of black holes by boundary measurements

Let

$$Lu = 0 \tag{5.1}$$

be the equation (1.1) in the cylinder $\mathbb{R} \times \mathcal{D}$, where $\mathcal{D} \subseteq \mathbb{R}^2$ is a bounded domain with smooth boundary $\partial\mathcal{D}$ such that the ergoregion $\Omega = \{g_{00}(x) < 0\}$ is contained inside \mathcal{D} . Consider the initial-boundary value problem for (5.1) in $\mathbb{R} \times \mathcal{D}$ with the boundary and initial conditions

$$u|_{\mathbb{R} \times \partial\mathcal{D}} = f, \tag{5.2}$$

$$u = 0 \text{ for } x_0 \ll 0, \ x \in \mathcal{D}, \tag{5.3}$$

where f has compact support in $\mathbb{R} \times \mathcal{D}$. Let Λ be the Dirichlet-to-Neumann (DN) operator on $\mathbb{R} \times \partial\mathcal{D}$, i.e.

$$\Lambda f = \sum_{j,k=0}^n g^{jk}(x) \frac{\partial u}{\partial x_j} \nu_k(x) \left(\sum_{p,r=0}^n g^{pr}(x) \nu_p(x) \nu_r(x) \right)^{-\frac{1}{2}} \Big|_{\mathbb{R} \times \partial\mathcal{D}}, \tag{5.4}$$

where $n = 2$, $\nu = (\nu_1, \nu_2)$ is the outward unit normal to $\partial\mathcal{D}$, and $u = u(x_0, x)$ is the solution of (5.1), (5.2), (5.3).

Let Γ be any open subset of $\partial\mathcal{D}$. We say that boundary measurements are performed on $\mathbb{R} \times \Gamma$ if we are able to measure the restriction $\Lambda f|_{\mathbb{R} \times \Gamma}$ for any smooth input f with support in $\mathbb{R} \times \bar{\Gamma}$.

Let $x' = \phi(x)$ be a diffeomorphism of $\bar{\mathcal{D}}$ onto $\bar{\mathcal{D}}$ such that $\phi(x) = x$ on Γ . Let $a(x) \in C^\infty(\bar{\Omega})$ be such that $a(x) = 0$ on Γ . It is well-known that if we make a change of variable

$$x' = \phi(x), \quad x_0' = x_0 + a(x), \tag{5.5}$$

then in coordinates (x_0', x') we get an initial-boundary value problem similar to (5.1), (5.2), (5.3) such that

$$\Lambda f|_{\mathbb{R} \times \Gamma} = \Lambda' f|_{\mathbb{R} \times \Gamma}, \tag{5.6}$$

for all f with support in $\mathbb{R} \times \bar{\Gamma}$, where Λ' is the DN operator in (x_0', x') coordinates. Therefore we have to study the determination of the metric from boundary measurements on $\mathbb{R} \times \Gamma$ only modulo changes of variables of the form (5.5). It was proven in [Esk10b] for $n \geq 2$ that boundary measurements on $\mathbb{R} \times \Gamma$ allow recovery of the ergosphere $\partial\Omega = \{g_{00} = 0\}$ and the metric on $\bar{\mathcal{D}} \setminus \Omega$ up to changes of variables (5.5).

It follows from the considerations in [Esk08] that if at least one point of $\partial\Omega$ is characteristic, then it is necessary to spend infinite time to recover the ergosphere $\partial\Omega$, i.e. for any T , boundary measurements on $[0, T] \times \Gamma$ do not determine $\partial\Omega$ in a neighborhood of the characteristic points. However, if all points of $\partial\Omega$ are not characteristic then there exists T_0 such that boundary measurements on $[0, T_0] \times \Gamma$ determine the ergosphere and the metric $[g_{jk}]_{j,k=0}^n$ on the ergosphere up to diffeomorphisms (5.5).

In this section, in the case $n = 2$, we expand upon the result of [Esk10b], treating the recovery of a black hole inside Ω . It follows from the results of section 3 that if $b_1(\theta) < 0$ (see (1.2), (1.3), (1.4)) and if $\partial\Omega$ is not characteristic then there exists a black hole Ω_0 inside Ω and the black hole event horizon $\partial\Omega_0$ is smooth.

Note that the equations for black holes depend only on the spatial part $G = \begin{bmatrix} g^{\rho\rho} & g^{\rho\theta} \\ g^{\theta\rho} & g^{\theta\theta} \end{bmatrix}$ of the inverse metric tensor $[g^{ij}]_{i,j=0}^2$. Introduce coordinates (ρ, θ) in $\overline{\Omega} \setminus \Omega_0$, $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, $0 \leq \rho \leq \rho_0(\theta)$ in $\overline{\Omega} \setminus \Omega_0$, extending those in section 2.2, so that $\rho = -\Delta$ near $\partial\Omega$ and $\rho = 0$ is the equation of $\partial\Omega$, and chosen so that the event horizon $\partial\Omega_0$ is a graph given by $\rho = \rho_0(\theta)$. Consider the equation for characteristics $\phi^\pm = \phi^\pm(\rho, \theta)$:

$$g^{\rho\rho}(\phi_\rho^\pm)^2 + 2g^{\rho\theta}\phi_\rho^\pm\phi_\theta^\pm + g^{\theta\theta}(\phi_\theta^\pm)^2 = 0. \tag{5.7}$$

It follows from (5.7) that the matrices $G(\rho, \theta)$ and $\lambda(\rho, \theta)G(\rho, \theta)$, where $\lambda(\rho, \theta) \neq 0$, produce the same characteristic equation, i.e. the characteristic equations (and black holes) do not depend on the scaling factor $\lambda(\rho, \theta)$.

Thus assuming that $g^{\rho\rho} \neq 0$ for all $0 \leq \rho \leq \rho_0$, we get

$$\phi_\rho^\pm = \frac{-g^{\rho\theta} \pm \sqrt{\rho_1}}{g^{\rho\rho}}\phi_\theta^\pm, \tag{5.8}$$

where we have used that $g^{\rho\rho}g^{\theta\theta} - (g^{\rho\theta})^2 = -\rho_1$, $\rho_1 = C^2\rho$. We impose the following boundary conditions on $\phi^+(\rho, \theta)$ and $\phi^-(\rho, \theta)$ when $\rho = 0$:

$$\phi^\pm(0, \theta) = \theta, \quad \theta \in [0, 2\pi]. \tag{5.9}$$

Consider the curves $\phi^+(\rho, \theta) = \theta_0$, $\phi^-(\rho, \theta) = \theta_0$ for fixed $\theta_0 \in \mathbb{R}/2\pi\mathbb{Z}$. It was shown in [Esk10] (see also section 3) that it is possible to use the time variable x_0 as a parameter for both curves. One of the curves, say $\phi^-(\rho, \theta) = \theta_0$, starts at $(0, \theta_0)$ when $x_0 = t_0$ and approaches the singularity at $0 \in \Omega_0$ when $x_0 \rightarrow +\infty$, crossing the event horizon $\partial\Omega_0$ at some time t . The second curve $\phi^+(\rho, \theta) = \theta_0$ ends at $(0, \theta_0)$ as x_0 increases. When $x_0 \rightarrow -\infty$, the curve $\phi^+(\rho, \theta) = \theta_0$ spirals around the event horizon $\partial\Omega$. For definiteness suppose $\phi^+ = \theta_0$ spirals counter-clockwise when $x_0 \rightarrow -\infty$. Note that

$$g^{\rho\rho}\phi_\rho^\pm + g^{\rho\theta}\phi_\theta^\pm = 0 \text{ at } (0, \theta_0). \tag{5.10}$$

Note that $\rho(\theta)$ is a periodic function on $(-\infty, \infty)$. Make the change of variables

$$\sigma = \phi^+(\rho, \theta), \quad \tau = \phi^-(\rho, \theta), \tag{5.11}$$

where $(\rho, \theta) \in \Pi$. The Jacobian is

$$\begin{aligned} \frac{\partial(\sigma, \tau)}{\partial(\rho, \theta)} &= \phi_\rho^+\phi_\theta^- - \phi_\rho^-\phi_\theta^+ = \frac{(-g^{\rho\theta} + \sqrt{\rho_1})\phi_\theta^+\phi_\theta^-}{g^{\rho\rho}} \\ &\quad - \frac{(-g^{\rho\theta} - \sqrt{\rho_1})\phi_\theta^-\phi_\theta^+}{g^{\rho\rho}} = \frac{2\sqrt{\rho_1}}{g^{\rho\rho}}\phi_\theta^+\phi_\theta^-. \end{aligned} \tag{5.12}$$

Therefore the map (5.11) is one-to-one when $\rho > 0$ and it is not smooth when $\rho = 0$. Note that $\phi^+(\rho, \theta) = \theta_0$ approaches $+\infty$ when $x_0 \rightarrow -\infty$. Make a new change of variables

$$y_1 = \frac{\sigma + \tau}{2} = \frac{\phi^+(\rho, \theta) + \phi^-(\rho, \theta)}{2}, \quad y_2 = \frac{\sigma - \tau}{2} = \frac{\phi^+(\rho, \theta) - \phi^-(\rho, \theta)}{2}. \tag{5.13}$$

Note that

$$y_1|_{\rho=0} = \theta, \quad y_2|_{\rho=0} = 0, \quad -\infty < \theta < \infty. \tag{5.14}$$

We have that $y_2 \rightarrow +\infty$ when $\sigma \rightarrow +\infty$. Denote the map (5.13) by Φ . Thus Φ maps Π onto the half-plane $\mathbb{R}^2 = \{-\infty < y_1 < \infty, y_2 > 0\}$. It follows from (5.14) that the map Φ is the identity on $\{\rho = 0\}$. Characteristic curves $\phi^+ = c_+, \phi^- = c_-$ become $y_1 + y_2 = c_+, y_1 - y_2 = c_-$ after the map Φ . Varying c_+, c_- we can fill the half-plane $\{y_1 \in \mathbb{R}, y_2 \geq 0\}$. Note that the event horizon $\rho = \rho_0(\theta)$ is the boundary of the strip Π . Note that the matrix G has the following form after applying the map Φ :

$$G' = \frac{1}{4} \hat{g}^{\sigma\tau} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \Phi G \Phi^t, \tag{5.15}$$

where

$$\hat{g}^{\sigma\tau} = g^{\rho\rho} \phi_\rho^+ \phi_\rho^- + g^{\rho\theta} (\phi_\theta^+ \phi_\rho^- + \phi_\rho^+ \phi_\theta^-) + g^{\theta\theta} \phi_\theta^+ \phi_\theta^- = \frac{-2C^2\rho}{g^{\rho\rho}} \phi_\theta^+ \phi_\theta^-, \tag{5.16}$$

since $\rho_1 = C^2\rho$.

Suppose we have another metric $[g_1^{jk}(\rho, \theta)]_{j,k=0}^2$ having the same boundary measurements on $\mathbb{R} \times \Gamma$. Then metrics g, g_1 are the same in $\mathcal{D} \setminus \Omega$ up to changes of variables (5.5) (cf [Esk10b]). Therefore we may assume that the ergosphere $\partial\Omega$ for both metrics is the same and the restriction of both metrics to $\partial\Omega$ is also the same. Suppose ϕ_1^\pm satisfy (5.8), (5.9) with g replaced by g_1 . Let $\phi_1^\pm(0, \theta) = \theta$ for all $\theta \in \mathbb{R}/2\pi\mathbb{Z}$. Make the change of variables $\sigma_1 = \phi_1^+(\rho, \theta), \tau_1 = \phi_1^-(\rho, \theta), \theta \in \mathbb{R}, 0 \leq \rho \leq \rho_0^{(1)}(\theta)$, where $\rho = \rho_0^{(1)}(\theta)$ is the equation of the event horizon $\partial\Omega'_0$. Make also the change of variables

$$y'_1 = \frac{\sigma_1 + \tau_1}{2}, \quad y'_2 = \frac{\sigma_1 - \tau_1}{2}. \tag{5.17}$$

Denote the map (5.17) by Φ_1 . Thus Φ_1 maps $\Pi' = \{\theta \in \mathbb{R}, 0 \leq \rho < \rho_0^{(1)}(\theta)\}$ onto $\mathbb{R}^2_+ = \{y_1 \in \mathbb{R}, y_2' \geq 0\}$. Note that Φ_1 is a homeomorphism and Φ_1 is the identity on $\{-\infty < \theta < \infty, \rho = 0\}$.

Let $G_1 = \begin{bmatrix} g_1^{\rho\rho} & g_1^{\rho\theta} \\ g_1^{\rho\theta} & g_1^{\theta\theta} \end{bmatrix}$ be the spacial part of the inverse metric tensor g_1 . Making the change of variables (5.17) we get analogously to (5.15)

$$G_1' = \frac{1}{4} \hat{g}_1^{\sigma_1\tau_1} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \Phi_1 G_1 \Phi_1^t, \tag{5.18}$$

where analogously to (5.16)

$$\hat{g}_1^{\sigma_1\tau_1} = \frac{-2C_1^2\rho}{g_1^{\rho\rho}} \phi_{1\theta}^+ \phi_{1\theta}^-, \quad C_1 > 0. \tag{5.19}$$

Combining (5.15) and (5.18) we get

$$G_1 = \lambda \Phi_1^{-1} \Phi G (\Phi_1^{-1} \Phi)^t \tag{5.20}$$

where

$$\lambda = \hat{g}_1^{\sigma_1\tau_1} (\hat{g}^{\sigma\tau})^{-1}. \tag{5.21}$$

It follows from (5.16), (5.19) that $\lambda \neq 0$ for $\rho \geq 0$ and smooth.

Analogously to (5.1) we have that $\frac{\partial(\sigma_1, \tau_1)}{\partial(\rho, \theta)} = \frac{2C_1\sqrt{\rho}}{g^{\rho\rho}}\phi_{1\theta}^+\phi_{1\theta}^-$. Thus $\frac{\partial(\sigma_1, \tau_1)}{\partial(\sigma, \tau)} = \frac{\partial(\sigma_1, \tau_1)}{\partial(\rho, \theta)} \left(\frac{\partial(\sigma, \tau)}{\partial(\rho, \theta)} \right)^{-1} \neq 0$ and smooth for $\rho \geq 0$.

Therefore the map $\Phi_1^{-1}\Phi$ of $\Pi = \{\theta \in \mathbb{R}, 0 \leq \rho < \rho_0(\theta)\}$ into $\Pi' = \{\theta \in \mathbb{R}, 0 \leq \rho < \rho_0'(\theta)\}$ is a diffeomorphism and $\Phi_1^{-1}\Phi$ is the identity on $\{\rho = 0, \theta \in \mathbb{R}\}$.

Taking the closure of Π and Π' we get a diffeomorphism of the event horizons $\partial\Omega_0$ and $\partial\Omega'_0$. Thus we have proven that the event horizon $\partial\Omega_0$ is determined uniquely up to diffeomorphism equal to the identity on the ergosphere.

References

- [BCO⁺11] Belgiorno F, Cacciatori S L, Ortenzi G, Rizzi L, Gorini V and Faccio D 2011 Dielectric black holes induced by a refractive index perturbation and the Hawking effect *Phys. Rev. D* **83** 024015
- [BLV⁺05] Barceló C, Liberati S, Visser M *et al* 2005 Analogue gravity *Living Rev. Relativ.* **8** 214
- [Esk08] Eskin G 2008 Optical Aharonov–Bohm effect: an inverse hyperbolic problems approach *Commun. Math. Phys.* **284** 317–43–839
- [Esk10] Eskin G 2010 Inverse hyperbolic problems and optical black holes *Commun. Math. Phys.* **297** 817–39
- [Esk10b] Eskin G 2010 Perturbations of the Kerr black hole and boundness of linear waves *J. Math. Phys.* **57** 112501
- [Esk14] Eskin G 2014 Nonstationary artificial black holes *Inverse Problems* **30** 125007
- [FFL⁺10] Fagnocchi S, Finazzi S, Liberati S, Kormos M and Trombettoni A 2010 Relativistic bose-einstein condensates: a new system for analogue models of gravity *New J. Phys.* **12** 095012
- [FN98] Frolov V and Novikov I 1998 *Black Hole Physics: Basic Concepts and New Developments* vol 96 (London: Springer/Kluwer)
- [Gor23] Gordon W 1923 Zur lichtfortpflanzung nach der relativittstheorie *Ann. Phys., Lpz.* **377** 421–56
- [Hal13] Hall M A 2013 *PhD Thesis* UCLA
- [LP00] Leonhardt U and Piwnicki P 2000 Relativistic effects of light in moving media with extremely low group velocity *Phys. Rev. Lett.* **84** 822
- [NVV02] Novello M, Visser M and Volovik G E 2002 *Artificial Black Holes* (Singapore: World Scientific)
- [PKR⁺08] Philbin T G, Kuklewicz C, Robertson S, Hill S, König F and Leonhardt U 2008 Fiber-optical analog of the event horizon *Science* **319** 1367–70
- [RMM⁺10] Rousseaux G, Maïssa P, Mathis C, Couillet P, Philbin T G and Leonhardt U 2010 Horizon effects with surface waves on moving water *New J. Phys.* **12** 095018
- [SU02] Schützhold R and Unruh W G 2002 Gravity wave analogues of black holes *Phys. Rev. D* **66** 044019
- [Unr81] Unruh W G 1981 Experimental black-hole evaporation? *Phys. Rev. Lett.* **46** 1351–3
- [Vis98] Visser M 1998 Acoustic black holes: horizons, ergospheres and Hawking radiation *Class. Quantum Gravity* **15** 1767–91
- [Vis12] Visser M 2012 *Survey of analogue spacetimes (Lecture Notes in Physics 870)* (Cham: Springer) pp 31–50
- [VMP10] Visser M and Molina-Paris C 2010 Acoustic geometry for general relativistic barotropic irrotational fluid flow *New J. Phys.* **12** 095014
- [Wal10] Wald R M 2010 *General Relativity* (Chicago, IL: University of Chicago Press)