Infinity in Mathematics and Philosophy

Philosophy 381
[day]s and [day]s, [time] – [time], August 22nd – December 10th, 2019
[room], University of Southern California

Instructors

Shieva Kleinschmidt
Department of Philosophy
Pronouns: 'she' or 'they' (singular)
Email: sjk@parthood.com
Website: www.parthood.com

Michael Hall
Department of Mathematics
Pronoun: 'he'
Email: hallma@usc.edu
https://sites.google.com/site/mikehallmath/

Office Hours
Stonier Hall 226
[days and times] and by appt.

Office Hours
Kaprielian Hall 447
[days and times] and by appt.

Course Description

Talk to a mathematician or a philosopher, and they'll tell you "things go weird at infinity". The concept of the infinite has been around for thousands of years, and it plays an important (and, in many cases, complicating) role throughout both Mathematics and Philosophy. In this class we will examine the notion and its strange and exciting implications and applications across a range of topics.

The course will be divided into 5 sections. First, we'll provide **an introduction** to the notion of the infinite: we'll give a brief overview of early thought about it, and then will contrast this with contemporary approaches, where the word 'infinity' is not used univocally across contexts.

Next we'll discuss the notion of infinite cardinals **in Mathematics**, and we'll endeavor to make them approachable by describing how they can be constructed and how we can prove things about them. We'll begin by describing the construction of the natural numbers, then the integers, rationals, and reals, before demonstrating how to construct infinite cardinals. We'll also cover ordinals, and show how they apply to infinity-involving cases such as Hercules and the Hydra, Hilbert's Hotel, and the Banach-Tarski Paradox.

In the third section of the course we'll examine the impact of the infinite on our theory of **reasoning and decision-making**. Typically, we calculate the expected utility of an action by adding together the expected value * probability of each possible outcome of the action. But what happens if some of the expected values are infinite? Is there a way to rescue this approach to decision-making in such contexts? We'll also cover the Two Envelope Paradox and the relationship between endorsement understood as precise credences and our ordinary notion of belief.

In the fourth part of the course we'll look at the role that the infinite plays in several areas of **Philosophy**. For instance, we'll examine infinite regresses, which are generated when some element stands in a relation that entails there's a further element, which itself stands in the same relation to yet a further element, and so on. Sometimes a series like this is welcome, and sometimes its postulation is treated as absurd. We'll examine how we might attempt to separate the two kinds of cases, and on what grounds. We'll also discuss supertasks and puzzles related to them: a supertask is a task with infinitely many steps completed in a finite amount of time. Again, though many of these are ordinary, everyday occurrences, some that we might describe give rise to paradox: in particular, we'll discuss Thomson's Lamp and Stoic Mixtures. Finally, we'll look at the infinitely small: we'll examine continuous models of regions, and will look at point-set topology in relation to objects, space, and time, and the philosophical puzzles of contact and motion that arise from those applications.

The final portion of the course will focus on the infinite **in Logic**. For instance, we will look at logics with not just two truth-values (true and false), and not just three truth-values (true, false, and neither), but *fuzzy* logics with infinitely many truth-values (intuitively, corresponding to degrees of truth). While we may be motivated by the Liar Paradox to opt for at least a 3-valued logic, we find motivation for fuzzy logics from vagueness paradoxes (such as the Sorites Paradox). We will also examine cardinality paradoxes, and will conclude our course (if we have time) with a discussion of the exciting and surprising Löwenheim-Skolem theorem.

This will be a challenging course. The class does not have any prerequisites (in Mathematics or in Philosophy), but it is a 300-level course that will move quickly through a number of challenging topics. If you have a background in Logic, Mathematics, or Computer Science, you may find that helpful.

Centrally, our hope is that in addition to giving you a thorough understanding of infinite cardinalities and their applications and implications, we will also provide a glimpse into a wide range of topics within Math, Logic, and Philosophy, providing you with a foundation for future exploration of those subjects.

Course Aspirations

- To give you basic understanding of the notion of infinite cardinalities, and the ways in which this notion has complicated and contributed to a wide variety of topics.

- To give you familiarity with a range of paradoxes involving infinity, and deep understanding of at least three of these, on which you will write exegetical papers.
- To give you a basis for future engagement with a range of topics in Mathematics and in Philosophy, regardless of your prior background with these fields.
- To increase your comfort with doing problem sets (if those are new to you) or with writing exegetical papers (if that is new to you).

Learning Objectives

- You will be able to describe (in symbols or in your own words) how to construct infinite cardinals.
- You will be able to do problem sets proving a variety of things about infinite cardinals.
- You will be able to, in your own words, present and explain proofs and paradoxes related to infinity, such as Cantor's Diagonal Proof, the Löwenheim-Skolem Theorem, and puzzles of contact from Philosophical Topology.
- You will be able to present and evaluate responses to a variety of paradoxes related to infinity.

Texts

There will be a variety of readings from textbooks and contemporary papers (see the course schedule below). In most cases, we will make chapters available to you electronically, and you will be able to find the papers online (in some cases these are available through JSTOR, which you can access through the university library's website). However, you will need to purchase the following:

- Enderton, *Elements of Set Theory* (Academic Press, 1977)
- Rayo, *On The Brink of Paradox: Highlights from the Intersection of Philosophy and Mathematics* (MIT Press, 2019).

Requirements

This course will include a variety of sorts of assignments, including problem sets for some of the more formal sections of the course, and several short papers as well. We will also have regular in-class activities, and depending on how class discussion goes, we may periodically assign impromptu additional, short out-of-class activities that will also count toward your in-class activities grade.

All assignments should be handed in to us in person, in class, on the date on which they are due (with the exception of the last paper, which can be submitted electronically by the deadline). If you miss an in-class assignment or are late with a paper or problem set, this can be excused if and only if (i) you have proof of an unavoidable medical situation, or (ii) you are facing a family emergency or a personal emergency. If you turn in a problem set or short paper late and you do not have a sufficient excuse of one of the two kinds just noted,

you will lose 5% of your assignment grade for every 24 hours that it is late. If you turn in your final draft late and you do not have a sufficient excuse, you will lose 10% of your assignment grade for every 24 hours that it is late. If you miss an in-class assignment without a sufficient excuse, the assignment cannot be made up – though we will drop the 4 lowest in-class assignment grades for every student, no questions asked, just to give you some wiggle room because we understand that life is complicated.

The course requirements will be graded are as follows. There are more details about each requirement below in the following sections.

Requirement	Weight	Due
In-Class Activities	30%	Most meetings, in person, as assigned
Problem Sets	30%	Due 1 week after assigned
Short Papers (3)	30%	Sep. 19th, Oct 24th, Nov 28th
Polished Final Draft	10%	Dec. 10th, 10am

1. In-Class Activities (30%)

Active participation in this class is an important component of your experience in this course. We will regularly have a range of in-class activities. Sometimes these will take the form of in-class quizzes (which will either test that you have done the reading, in which case the answers will be *very* easy to answer if, and only if, you've actually done the reading, or they will test comprehension of that day's lecture, in which case you can use your notes from that day). Sometimes, instead, these will take the form of short writing assignments, or short group assignments. Sometimes, these assignments will take the form of problems that resemble those you've found (or will shortly find) in your take-home problem sets. And sometimes, if something comes up in class and an out-of-class assignment seems that it will be particularly helpful in furthering student learning and engagement with the topics, these will have an out-of-class component (such as watching a movie, writing a question in preparation for the next lecture, etc).

These activities are intended to provide more structured, regular engagement with the course content, and will also give us feedback on the extent to which you're understanding the material and which things we might provide more help on.

2. Problem Sets (30%)

Many meetings of our course will involve working through problems and proofs, and at several points you will be assigned readings with problem sets as well. We will assign you problems from the texts and elsewhere, and you will be responsible for turning these in to us, in person, at the following class meeting. (Unlike papers, you need only give us one copy of your problem sets.)

In all cases, we will expect you to show your work. For many problems, even if you end up with the wrong answer, the things you've tried will count for a large amount of the credit. Handwritten work is welcomed, but please make your work legible.

3. Short Papers (30%)

Over the course of the semester, you will write 3 short (6-7 page, double-spaced, 12 pt. in Times New Roman) exegetical papers, each worth 10% of your final grade. The first of these papers will be on a topic we covered in our Introduction To Infinity. The second paper will be on a topic either from Infinity in Mathematics, or Infinity in Value and Reasoning. The third paper will be on a topic either from Infinity in Philosophy, or Infinity in Logic.

These papers will be exegetical: you will not be required to think of new, original ideas for the papers, but rather, the point is for you to practice explaining something challenging in your own words, and in a way that makes it accessible and understandable to a person with no prior knowledge about the topic.

You are welcome to choose your own topics within the areas noted above, and if there is, e.g., something related to Infinity in Mathematics or Infinity in Philosophy that we aren't covering in the course and that you would like to write on, you are welcome to come to us for approval of the topic. We encourage you to explore beyond the bounds of our course (it is, after all, a course about the boundless).

The short papers you turn in to us should be polished, and should include all relevant citations. This is to say: *any time you present an idea that is not yours*, you must cite the text from which you got it, including the page number of where precisely it came from.

Finally, these papers should be handed in to us in person, in class, on or before the deadline. Please bring two hard copies of each paper.

3. Final Draft (10%)

During the semester you will choose one of your papers that can be further developed and polished. You will expand this to a 10-page (double-spaced, 12 pt. in Times New Roman) paper that addresses any concerns raised about the short version, and that goes more in-depth into the topic you covered.

Our intention with requiring these longer, final drafts is (i) to give you a chance to engage more deeply with whichever paper topic you found most engaging, and (ii) to give you practice in revising and improving your writing.

These final drafts will be due at the end of the time scheduled for our class's final exam. Unlike the Short Papers, these Final Drafts can be submitted to us via email as either PDFs or docs.

Contacting Us

Please put "Infinity Course" in the subject line of any email you send about our course. In general, we can be expected to respond within 72 hours of receiving an email (not including weekends and holidays).

Class Conduct

Behavior that hinders the learning of your classmates is not allowed. Thus, cell phones must be turned off during class (or at least, set to silent and not used during class). And using computers for anything that is not course-related is not allowed. (E.g., no checking Facebook during lectures.)   If we notice you engaging in these or other disruptive behaviors, you will be asked to leave the classroom.

Equality, Diversity, and Support

This classroom is a safe environment. Any discrimination on the basis of race, gender, sex, sexuality, socioeconomic status, disability, national origin, religion, or age will not be tolerated. If at any time while at USC you feel you have experienced harassment or discrimination, you can file a complaint through the Office of Equity and Diversity (OED)/Title IX Compliance – call (213) 740-5086, or visit http://equity.usc.edu for more information. You are also welcome to bring the complaint to any faculty or staff member at USC.

Statements for Students with Disabilities

Any student requesting academic accommodations based on a disability is required to register with Disability Services and Programs (DSP) each semester. A letter of verification for approved accommodations can be obtained from DSP. Please be sure the letter is delivered to us as early in the semester as possible. DSP is located in STU 301 and is open 8:30 a.m. - 5:00 p.m., Monday through Friday. The phone number for DSP is (213) 740-0776.

Relatedly: We have a general policy of allowing food in our classes, in part to create a more informal environment. But: we understand that it is not uncommon for people to be very sensitive to sounds like crunching or plastic rustling, or to smells, and for them, allowing food would make the class less accessible. So: if this applies to you, contact us (anonymously if you like, perhaps via a note in either of our mailboxes) letting us know, and we will disallow food during class for the semester.

Academic Conduct and Support Systems

Academic Conduct:
Plagiarism – presenting someone else's ideas as your own, either verbatim or recast in your own words – is a serious academic offense with serious consequences. Please familiarize yourself with the discussion of plagiarism in *SCampus* in Part B, Section 11, "Behavior Violating University Standards" policy.usc.edu/scampus-part-b. Other forms of academic dishonesty are equally unacceptable. See additional information in *SCampus* and university policies on scientific misconduct, http://policy.usc.edu/scientific-misconduct.

Support Systems:
Student Counseling Services (SCS) – (213) 740-7711 – 24/7 on call
Free and confidential mental health treatment for students, including short-term psychotherapy, group counseling, stress fitness workshops, and crisis intervention. engemannshc.usc.edu/counseling

National Suicide Prevention Lifeline – 1 (800) 273-8255
Provides free and confidential emotional support to people in suicidal crisis or emotional distress 24 hours a day, 7 days a week. www.suicidepreventionlifeline.org

Relationship and Sexual Violence Prevention Services (RSVP) – (213) 740-4900 – 24/7 on call
Free and confidential therapy services, workshops, and training for situations related to gender-based harm. engemannshc.usc.edu/rsvp

Sexual Assault Resource Center
For more information about how to get help or help a survivor, rights, reporting options, and additional resources, visit the website: sarc.usc.edu

Bias Assessment Response and Support
Incidents of bias, hate crimes and microaggressions need to be reported allowing for appropriate investigation and response. studentaffairs.usc.edu/bias-assessment-response-support

Student Support and Advocacy – (213) 821-4710
Assists students and families in resolving complex issues adversely affecting their success as a student EX: personal, financial, and academic. studentaffairs.usc.edu/ssa

Diversity at USC
Information on events, programs and training, the Diversity Task Force (including representatives for each school), chronology, participation, and various resources for students. diversity.usc.edu

USC Emergency Information
Provides safety and other updates, including ways in which instruction will be continued if an officially declared emergency makes travel to campus infeasible. emergency.usc.edu

USC Department of Public Safety – UPC: (213) 740-4321 – HSC: (323) 442-1000 – 24-hour emergency or to report a crime.
Provides overall safety to USC community. dps.usc.edu

Schedule of Topics

Readings with a double asterisk (**) are optional. Topics and readings may change: We will update you via email of any changes.

An Introduction to the Infinite

August   22   Overview of the Infinite, and Overview of Course Logistics

August   24   A Quick History of the Concept
               Rucker, *Infinity and the Mind*, ch. 1

August   29   Zeno's Paradoxes
               Huggett, "Zeno's Paradoxes", sect. 3.1, 3.2, 3.3 (Stanford Encyc. of Philosophy)

August   31   Contemporary Approaches: Many Concepts

Infinity in Mathematics

September 5   Constructing The Natural Numbers
               Enderton, *Elements of Set Theory*, ch. 4

September 7   Constructing The Real Numbers
               Enderton, *Elements of Set Theory*, ch. 5
               — **Problem Set Due** —

September 12   Constructing The Real Numbers, Continued
               — **Paper 1 due** —

September 14   Constructing Infinite Cardinals
               Enderton, *Elements of Set Theory*, ch. 6
               **Rayo, *On The Brink of Paradox*, ch. 1
               — **Problem Set Due** —

September 19   Different Levels of Infinity
               Rayo, *On The Brink of Paradox*, Ch. 2

September 21   Ordinals
               Enderton, *Elements of Set Theory*, ch. 7
               — **Problem Set Due** —

September 26   The Puzzle of Hercules and the Hydra
  **Kirby and Paris, "[Accessible Independence Results for Peano Arithmetic](#)"
    (Bulletin of the London Mathematical Society, 1982)

September 28   From Hilbert's Hotel to The Banach-Tarski Theorem
  Rayo, *On The Brink of Paradox*, Ch. 8
  **Gamow, *One, Two, Three ... Infinity: Facts and Speculations of Science*, Ch. 1,
    pp. 3-23
  **Pires, *[Hospitality at the Hilbert Hotel: How Big is Infinity?](#), 2016*

October   3   The Banach-Tarski Theorem, Continued

Infinity in Reasoning

October   5   Expected Utility Calculations – Introduction
  — **Problem Set Due** —

October   10   Infinite Value and Expected Utility
  Nover and Hajek, "[Vexing Expectations](#)" (*Mind* 2004 – via JSTOR)
  **Hansen, "[Double Up on Heaven](#)" (*Thought* 2015)

October   12   The Two Envelope Paradox
  Arntzenius and McCarthy, "[The Two Envelope Paradox and Infinite
    Expectations](#)", (Analysis 1997 – via JSTOR)

October   17   Precise Credences vs. Beliefs

Infinity in Philosophy

October   19   Infinite Regresses: Overview
  Cameron, "[Infinite Regress Arguments](#)" (Stanford Encyc. of Philosophy)

October   24   Infinite Regresses: When Are They Vicious?
  — **Paper 2 due** —

October   26   Supertask Puzzles: Thomson's Lamp and Stoic Mixtures
  Manchak and Roberts, "[Supertasks](#)" (Stanford Encyclopedia of Philosophy)
  **Thomson, "[Tasks and Super-Tasks](#)" (*Analysis*, 1954 – via JSTOR)

October   31   Discrete vs. Continuous Accounts of Space
  Forrest, "[From Ontology to Topology in the Theory of Regions](#)" (*The Monist*, via
    JSTOR)

November 2   Philosophical Topology: Introduction

Callender and Weingard, "An Introduction to Topology" (*The Monist*, via JSTOR)

November 7   Philosophical Topology: Puzzles of Contact
　　　　　　Hudson, *The Metaphysics of Hyperspace*, ch. 3

Infinity in Logic

November 9   Cardinality Paradoxes – Introduced
　　　　　　Uzquiano, "Recombination and Paradox" (*Philosophers' Imprint*, 2015)

November 14   Cardinality Paradoxes – Continued

November 16   Many-Valued Logics
　　　　　　Bergmann, *An Introduction to Many-Valued and Fuzzy Logic*, ch. 5

November 21   Fuzzy (infinite-valued) Logics
　　　　　　— **Problem Set Due** —
　　　　　　Bergmann, *An Introduction to Many-Valued and Fuzzy Logic*, ch. 13

November 23   No Class – Thanksgiving Holiday

November 28   The Löwenheim-Skolem Theorem – Walking Through the Proof
　　　　　　**Suber, "The Löwenheim-Skolem Theorem"
　　　　　　— **Problem Set Due** —

November 30   The Löwenheim-Skolem Theorem – Implications
　　　　　　Bays, "Skolem's Paradox" (Stanford Encyclopedia of Philosophy)
　　　　　　— **Paper 3 due** —

December 10   10am: **Revised Paper due**, submit to both instructors via email