

Homework 6 Solutions

1. Suppose a particle of mass m moves along a twice-differentiable path $\vec{x}(t)$ satisfying the differential equation

$$m\vec{x}''(t) = -\nabla V(\vec{x}(t)),$$

where $V: \mathbb{R}^n \rightarrow \mathbb{R}$ is a differentiable function. Let

$$E(t) = \frac{1}{2}m|\vec{x}'(t)|^2 + V(\vec{x}(t)).$$

Prove that $E(t)$ is constant. (This is the standard conservation of energy law for motion under the influence of a potential V .)

Solution:

Writing $|\vec{x}'(t)|^2 = \vec{x}'(t) \cdot \vec{x}'(t)$, and using the product rule to differentiate the first term, and chain rule for the second, we have

$$\frac{d}{dt}E(t) = m\vec{x}''(t) \cdot \vec{x}'(t) + \nabla V(\vec{x}(t)) \cdot \vec{x}'(t).$$

Since $m\vec{x}''(t) = -\nabla V(\vec{x}(t))$, we get

$$\frac{d}{dt}E(t) = -\nabla V(\vec{x}(t)) \cdot \vec{x}'(t) + \nabla V(\vec{x}(t)) \cdot \vec{x}'(t) = 0,$$

so $E(t)$ is constant.

2. Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ be a differentiable function. Let us write $f = f(u, v)$, and let $g(x, y) = f(x + y, x - y)$. Prove that

$$\frac{\partial g}{\partial x} \frac{\partial g}{\partial y} = \left(\frac{\partial f}{\partial u}\right)^2 - \left(\frac{\partial f}{\partial v}\right)^2.$$

Solution:

The functions f and g are related by the change of variables $u = x + y$, $v = x - y$. We calculate

$$\begin{aligned} \frac{\partial u}{\partial x} &= 1, & \frac{\partial u}{\partial y} &= 1 \\ \frac{\partial v}{\partial x} &= 1, & \frac{\partial v}{\partial y} &= -1. \end{aligned}$$

By the chain rule, we have

$$\begin{aligned}\frac{\partial g}{\partial x} &= \frac{\partial u}{\partial x} \frac{\partial f}{\partial u} + \frac{\partial v}{\partial x} \frac{\partial f}{\partial v} = \frac{\partial f}{\partial u} + \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial y} &= \frac{\partial u}{\partial y} \frac{\partial f}{\partial u} + \frac{\partial v}{\partial y} \frac{\partial f}{\partial v} = \frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}.\end{aligned}$$

Multiplying the above we get

$$\frac{\partial g}{\partial x} \frac{\partial g}{\partial y} = \left(\frac{\partial f}{\partial u} \right)^2 - \left(\frac{\partial f}{\partial v} \right)^2$$

as desired.

3. Show that at any point where the function $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ is smooth, we have the identity

$$\left(\frac{\partial f}{\partial x} \right)^2 + \left(\frac{\partial f}{\partial y} \right)^2 = \left(\frac{\partial g}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial g}{\partial \theta} \right)^2$$

where $g(r, \theta) = f(r \cos \theta, r \sin \theta)$, (x, y) are standard Cartesian coordinates, and (r, θ) are polar coordinates.

Solution:

By the chain rule, we have

$$\begin{aligned}\frac{\partial g}{\partial r} &= \frac{\partial x}{\partial r} \frac{\partial f}{\partial x} + \frac{\partial y}{\partial r} \frac{\partial f}{\partial y} = (\cos \theta) \frac{\partial f}{\partial x} + (\sin \theta) \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial \theta} &= \frac{\partial x}{\partial \theta} \frac{\partial f}{\partial x} + \frac{\partial y}{\partial \theta} \frac{\partial f}{\partial y} = (-r \sin \theta) \frac{\partial f}{\partial x} + (r \cos \theta) \frac{\partial f}{\partial y},\end{aligned}$$

so

$$\begin{aligned}\left(\frac{\partial g}{\partial r} \right)^2 &= \cos^2 \theta \left(\frac{\partial f}{\partial x} \right)^2 + 2 \sin \theta \cos \theta \left(\frac{\partial f}{\partial x} \right) \left(\frac{\partial f}{\partial y} \right) + \sin^2 \theta \left(\frac{\partial f}{\partial y} \right)^2 \\ \frac{1}{r^2} \left(\frac{\partial g}{\partial \theta} \right)^2 &= \sin^2 \theta \left(\frac{\partial f}{\partial x} \right)^2 - 2 \sin \theta \cos \theta \left(\frac{\partial f}{\partial x} \right) \left(\frac{\partial f}{\partial y} \right) + \cos^2 \theta \left(\frac{\partial f}{\partial y} \right)^2.\end{aligned}$$

Using $\sin^2 \theta + \cos^2 \theta = 1$, we get

$$\left(\frac{\partial g}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial g}{\partial \theta} \right)^2 = \left(\frac{\partial f}{\partial x} \right)^2 + \left(\frac{\partial f}{\partial y} \right)^2$$

as desired.

4. Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ be the function $f(x, y)$ which satisfies

$$f(r \cos \theta, r \sin \theta) = r + \sin(\theta), \quad r \neq 0.$$

Using the result of the previous problem, find the maximum value of $D_{\vec{u}} f(2, 0)$, where \vec{u} is an arbitrary unit vector.

(Hint: What can you say about $|\nabla f(2, 0)|$?)

Solution: The maximum value of $D_{\vec{u}}f(2, 0)$ is equal to $|\nabla f(2, 0)|$. Using the previous problem we calculate

$$|\nabla f|^2 = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = \left(\frac{\partial g}{\partial r}\right)^2 + \frac{1}{r^2} \left(\frac{\partial g}{\partial \theta}\right)^2 = 1 + \frac{1}{r^2} \cos^2 \theta.$$

At the point $(2, 0)$ we have $r = 2$, $\cos \theta = 1$, so

$$|\nabla f(2, 0)|^2 = 1 + \frac{1}{4}(1)^2 = \frac{5}{4}.$$

Thus the maximum value of $D_{\vec{u}}f(2, 0)$ is equal to $\frac{\sqrt{5}}{2}$.